Úkol-2021
XI - 1

Moment hybnosti

· přednáška - definice, komutátor $[L_i, L_j] = \varepsilon_{ijk} L_z i\hbar$

$[L^2, L_i] = 0$

· cvičení 3.3. :
matice pro $l = 1/2$
transformace báze → diag matice
⋮

· 2D rotátor L^2, L_z, L_+, L_- ~~jako~~ akce na stavy $|l, m\rangle$
· nalezení hodnoty l, m L_+, L_- na $|1 0\rangle$ stav
· sférické harmoniky

· 1D rotor $\frac{1}{\sqrt{2\pi}} e^{i\phi m}$ zadání

→ trochu komutátorů, rozklad, čas. vývoj, ? vl. fce L^2, L_z, komutátory jako matice?
sym., integrace, ověřit normalizaci, OG, matrix repre, matrix elem $x, x^2, \dots$

- $[L_x, x]$, $[L_y, x]$, $[L_z, x]$ $L_x = y p_z - z p_y$, $L_y = z p_x - x p_z$, $L_z = x p_y - y p_x$

$$[L_x, x] = [y p_z - z p_y, x] = p_x \underset{\hookrightarrow 0}{[y, x]} - p_y \underset{\hookrightarrow 0}{[z, x]} = 0$$

$$[x, p_x] = i\hbar$$

$$[L_y, x] = [z p_x - x p_z, x] = z[p_x, x] = -i\hbar z$$

$$[L_z, x] = [x p_y - y p_x, x] = -\underbrace{[p_x, x]}_{-i\hbar} y = i\hbar y$$

- Mějme částici ve stavu $\psi = N(\cos\theta + 1)\sin\theta\, e^{i\phi}$, určete N

$$\langle\psi|\psi\rangle = N^2 \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin\theta \left[(\cos\theta + 1)\sin\theta\, e^{-i\phi}\right]\left[(\cos\theta + 1)\sin\theta\, e^{i\phi}\right]$$

$\underset{\to 1}{}$

$$= N^2 2\pi \int_0^{\pi} \sin^3\theta \left[\cos^2\theta + 2\cos\theta + 1\right] d\theta$$

$$= N^2 2\pi \int_0^{\pi} \sin\theta (1 - \cos^2\theta)(\cos^2\theta + 2\cos\theta + 1)\, d\theta$$

$\cos\theta = t$
$-d\theta \sin\theta = dt$
$\cos\pi = -1$
$\cos 0 = 1$

$$= -N^2 2\pi \int_1^{-1} dt\, (1 - t^2)(t^2 + 2t + 1)$$

$$= -N^2 2\pi \int_1^{-1} dt \left(-t^4 - 2t^3 - \cancel{t^2} + \cancel{t^2} + 2t + 1\right)$$

$$= N^2 2\pi \int_1^{-1} \left[-\frac{t^5}{5}\Big|_1^{-1} - \frac{2t^4}{4}\Big|_1^{-1} + \frac{2t^2}{2}\Big|_1^{-1} + t\Big|_1^{-1}\right] \cdot (-1)$$ ← from dt

$$= N^2 2\pi \left[\frac{2}{5} - 0 + 0 - 2\right] = 2\pi N^2 \frac{8}{5} = 1$$

$$N^2 = \frac{5}{16\pi}$$

$$N = \sqrt{\frac{5}{16\pi}}$$ OK

Mějme $Y_1^1 = -\sqrt{\frac{3}{8\pi}} \sin\theta e^{i\phi}$, $Y_2^1 = \sqrt{\frac{15}{8\pi}} \cos\theta \sin\theta e^{i\phi}$

ověřit N

$$\langle Y_1^1 | Y_1^1 \rangle = \int_0^{\pi} d\theta \sin\theta \int_0^{2\pi} d\phi \left[-\sqrt{\frac{3}{8\pi}} \sin\theta \underbrace{e^{-i\phi} \right] \left[-\sqrt{\frac{3}{8\pi}} \sin\theta e^{i\phi}}_{=1} \right]$$

$$= \int_0^{\pi} d\theta \sin\theta \cdot 2\pi \cdot \frac{3}{8\pi} \underbrace{\sin^2\theta}_{(1-\cos^2\theta)} \qquad \begin{matrix} \cos\theta = t \\ -\sin\theta d\theta = dt \end{matrix}$$

$$= -\int_{+1}^{-1} \frac{3}{4} (1-t^2) dt$$

$$= \int_{-1}^{1} \frac{3}{4} \int_{-1}^{1} (1-t^2) dt$$

$$= \frac{3}{4} \left[t \Big|_{-1}^{1} - \frac{t^3}{3} \Big|_{-1}^{1} \right] = \frac{3}{4} \left[2 - \frac{2}{3} \right] = \frac{3}{4} \cdot \frac{4}{3} = 1$$

$$\langle Y_2^1 | Y_2^1 \rangle = \frac{15}{8\pi} \int_0^{\pi} d\theta \sin\theta \underbrace{\int_0^{2\pi} d\phi}_{2\pi} \cos^2\theta \underbrace{\sin^2\theta}_{(1-\cos^2\theta)} \cancel{e^{i\phi}} \cdot \cancel{e^{-i\phi}}$$

$$= \frac{15 \cdot 2\pi}{8\pi} \int_0^{\pi} d\theta \sin\theta \left[\cos^2\theta - \cos^4\theta \right] \qquad \begin{matrix} \cos\theta = t \\ -\sin\theta d\theta = dt \end{matrix}$$

$$= \frac{15 \cdot 2\pi}{8\pi} \int_{-1}^{1} dt \left[t^2 - t^4 \right]$$

$$= \frac{15 \cdot 2\pi}{8\pi} \left[\frac{t^3}{3} \Big|_{-1}^{1} - \frac{t^5}{5} \Big|_{-1}^{1} \right] = \frac{15 \cdot 2\pi}{8\pi} \left[\frac{2}{3} - \frac{2}{5} \right] = \frac{15}{8\pi} \left[\frac{10-6}{15} \right] \cdot 2\pi = 1$$

$$\langle Y_1^1 | Y_2^1 \rangle = \frac{3}{8\pi} \sqrt{5} \int_0^{\pi} d\theta \sin\theta \underbrace{\int_0^{2\pi} d\phi}_{\to 2\pi} \sin^2\theta \cos\theta \underbrace{e^{-i\phi} e^{i\phi}}_{\to 1}$$

$$= \frac{3\sqrt{5}}{4\pi} \int_0^{\pi} d\theta \sin\theta (1-\cos^2\theta) \cos\theta$$

$$= \frac{3\sqrt{5}}{4\pi} \int_{-1}^{1} dt (1-t^2) t$$

$$= \frac{3\sqrt{5}}{4\pi} \int_{-1}^{1} dt \underbrace{(t - t^3)}$$ liché fce $\to \int = 0$

$\to$ kolmé

ÚKM-2011
XI-3

$$Y_0^0 = \frac{1}{\sqrt{4\pi}} \qquad Y_1^0 = \sqrt{\frac{3}{4\pi}} \cos\theta$$

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(|Y_0^0\rangle + |Y_1^0\rangle\right), \quad \langle L^2\rangle, \quad \langle L_z\rangle$$

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{4\pi}} + \sqrt{\frac{3}{4\pi}}\cos\theta\right) = \frac{1}{\sqrt{8\pi}}\left(1 + \sqrt{3}\cos\theta\right)$$

$$\langle L^2\rangle = \frac{1}{\sqrt{2}}\left(\langle Y_0^0| + \langle Y_1^0|\right) L^2 \frac{1}{\sqrt{2}}\left(|Y_0^0\rangle + |Y_1^0\rangle\right)$$

$$L^2 |Y_\ell^m\rangle = \hbar^2 \ell(\ell+1)|Y_\ell^m\rangle$$

$$= \frac{1}{2}\left(\langle Y_0^0| + \langle Y_1^0|\right)\left(0\,|Y_0^0\rangle + 2\hbar^2 |Y_1^0\rangle\right)$$

$$= \frac{1}{2}\,2\hbar^2 = \hbar^2 \quad \text{(average of eigenvals)}$$

$$\langle L_z\rangle = \frac{1}{\sqrt{2}}\left(\langle Y_0^0| + \langle Y_1^0|\right) \underbrace{L_z \frac{1}{\sqrt{2}}}\left(|Y_0^0\rangle + |Y_1^0\rangle\right)$$

$$L_z |Y_\ell^m\rangle = \hbar m |Y_\ell^m\rangle$$

using θ, ϕ

$$L_z = -i\hbar\frac{\partial}{\partial\phi} \qquad L^2 = -\hbar^2\left(\frac{\partial^2}{\partial\theta^2} + \frac{\cos\theta}{\sin\theta}\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right)$$

$$-i\hbar\frac{\partial}{\partial\phi}\cdot\frac{1}{\sqrt{8\pi}}\left(1 + \sqrt{3}\cos\theta\right) = 0$$

$$-\hbar^2\left(\frac{\partial^2}{\partial\theta^2} + \frac{\cos\theta}{\sin\theta}\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right)\frac{1}{\sqrt{8\pi}}\left(1 + \sqrt{3}\cos\theta\right)$$

$$= -\hbar^2\frac{\sqrt{3}}{\sqrt{8\pi}}\left(\frac{\partial^2}{\partial\theta^2} + \frac{\cos\theta}{\sin\theta}\frac{\partial}{\partial\theta}\right)(\cos\theta)$$

$$= -\hbar^2\frac{\sqrt{3}}{\sqrt{8\pi}}\left(-\cos^2\theta + \frac{\cos\theta}{\sin\theta}(-\sin\theta)\right) = \hbar^2\sqrt{\frac{3}{8\pi}}\left(2\cos\theta + \cancel{\cos^2\theta}\right)$$

$$\langle L^2\rangle = \frac{1}{\sqrt{8\pi}}\hbar^2\sqrt{\frac{3}{8\pi}}\int_0^\pi d\theta \sin\theta \underbrace{\int_0^{2\pi} d\phi}_{2\pi}\left[1 + \sqrt{3}\cos\theta\right]\left[2\cos\theta + \cancel{\cos^2\theta}\right]$$

$$= \frac{\sqrt{3}}{8\pi}\hbar^2\,2\pi\int_0^\pi d\theta\,\sin\theta\left[\cos\theta + \cancel{\cos^2\theta} + 2\sqrt{3}\cos^2\theta + \cancel{\sqrt{3}\cos^3\theta}\right]$$

$$= \cancel{\frac{\sqrt{3}}{8\pi}\hbar^2}\ \frac{\sqrt{3}}{4}\hbar^2\int_{-1}^{1} dt\left[\underset{\to 0}{t} + 2t^2(\cancel{1}\sqrt{3}) + \underset{\to 0}{\cancel{\sqrt{3}t^3}}\right]$$

$$= \frac{\sqrt{3}}{4}\hbar^2\left[\frac{t^3}{3}\Big|_{-1}^{1}(\cancel{4}\,2\sqrt{3})\right] = \frac{\sqrt{3}}{4}\hbar^2\,\frac{2}{3}(\cancel{4}\,2\sqrt{3}) = \frac{\hbar^2}{2}\frac{(2\sqrt{3})}{\sqrt{3}} = \hbar^2 \quad \text{OK}$$

$$[L_x, r^2] = [L_x, x^2+y^2+z^2] = [yp_z - zp_y, x^2+y^2+z^2]$$

$$= [yp_z, z^2] - [zp_y, y^2] = y[p_z, z^2] - z[p_y, y^2]$$

$$= y[p_z z^2 - z^2 p_z] - z[p_y y^2 - y^2 p_y]$$

$$= y[p_z z^2 - z p_z z + z p_z z - z^2 p_z]$$

$$- z[p_y y^2 - y p_y y + y p_y y - y^2 p_y]$$

$$= y\{\underset{-i\hbar}{[p_z, z]} z + z \underset{-i\hbar}{[p_z, z]}\} - z\{\underset{-i\hbar}{[p_y, y]} y + y \underset{-i\hbar}{[p_y, y]}\}$$

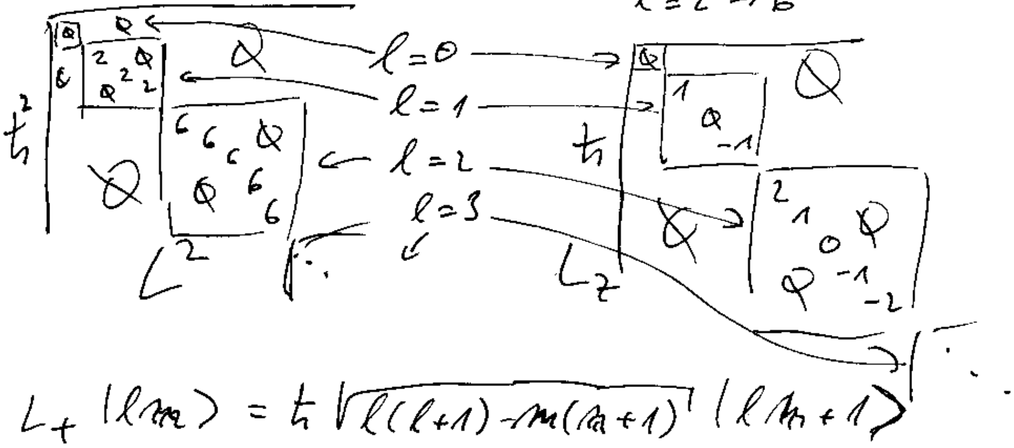
$= ~~i\hbar [-y z - z~~ $ $= -i\hbar [2yz - 2zy]$

$= 0$

Matrix

$$L^2 |lm\rangle = \hbar^2 l(l+1) |lm\rangle$$

$l = 0 \to 0$
$l = 1 \to 2$
$l = 2 \to 6$



$$L_+ |lm\rangle = \hbar \sqrt{l(l+1) - m(m+1)}\, |l\, m+1\rangle$$

$l = 0$ $m = 0$
0

$l = 1$
$m = 1$ $\sqrt{1\cdot 2 - 1\cdot 2} = 0$
$m = 0$ $\sqrt{1\cdot 2 - 0} = \sqrt{2}$
-1 $m = -1$ $\sqrt{1\cdot 2 + 1(0)} = \sqrt{2}$

$l = 2$
$m = 1$ $\sqrt{2\cdot 3 - 1\cdot 2} = 2$
$m = 0$ $\sqrt{2\cdot 3 - 0} = \sqrt{6}$
$m = -1$ $\sqrt{2\cdot 3 + 1\cdot 0} = \sqrt{6}$
$m = -2$ $\sqrt{6 + 2(1)} = 2$

$1\ 0\, |lm\rangle$

$$\to \quad \langle l'm'| \quad L_+ = \begin{pmatrix} 0 & & \\ & \begin{matrix} 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} \\ 0 & & 0 \end{matrix} & \\ & & \begin{matrix} 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{6} & 0 & 0 \\ 0 & 0 & 0 & \sqrt{6} & 0 \\ 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{matrix} \end{pmatrix}$$

$$\langle l'm' | L_+ | lm \rangle$$

$$\langle l'm' | l\, m+1 \rangle = \delta_{ll'} \delta_{m'\, m+1}$$

$$\to L_+ \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \sqrt{2}\hbar \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad \text{v } S_{l'm'} \quad \begin{matrix} m = 1 \\ m = 0 \\ m = -1 \end{matrix}$$

$$L_- \begin{pmatrix} 0 & 0 & 0 \\ \sqrt{2} & 0 & 0 \end{pmatrix}$$

ÚKM-2021
XI-5

$L_+ = L_{\cancel{x}} + iL_y$

$L_- = L_x - iL_y$

$L_x = \frac{1}{2}(L_+ + L_-)$

$L_y = \frac{1}{2i}(L_+ - L_-)$

$$\rightarrow L_x = \frac{\hbar}{2}\left[\begin{pmatrix} 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ \sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \end{pmatrix}\right] = \frac{\hbar}{2}\begin{pmatrix} 0 & \sqrt{2} & 0 \\ \sqrt{2} & 0 & \sqrt{2} \\ 0 & \sqrt{2} & 0 \end{pmatrix}$$

$$L_y = \frac{\hbar}{2i}\begin{pmatrix} 0 & \sqrt{2} & 0 \\ -\sqrt{2} & 0 & \sqrt{2} \\ 0 & -\sqrt{2} & 0 \end{pmatrix}$$

$$L_x^2 = \frac{\hbar^2}{4}\begin{pmatrix} 0 & \sqrt{2} & 0 \\ \sqrt{2} & 0 & \sqrt{2} \\ 0 & \sqrt{2} & 0 \end{pmatrix}\begin{pmatrix} 0 & \sqrt{2} & 0 \\ \sqrt{2} & 0 & \sqrt{2} \\ 0 & \sqrt{2} & 0 \end{pmatrix} = \frac{\hbar^2}{4}\begin{pmatrix} 2 & 0 & 2 \\ 0 & 4 & 0 \\ 2 & 0 & 2 \end{pmatrix} = \hbar^2\begin{pmatrix} 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \end{pmatrix}$$

$$L_y^2 = -\frac{\hbar^2}{4}\begin{pmatrix} 0 & \sqrt{2} & 0 \\ -\sqrt{2} & 0 & \sqrt{2} \\ 0 & -\sqrt{2} & 0 \end{pmatrix}\begin{pmatrix} 0 & \sqrt{2} & 0 \\ -\sqrt{2} & 0 & \sqrt{2} \\ 0 & -\sqrt{2} & 0 \end{pmatrix} = -\frac{\hbar^2}{4}\begin{pmatrix} -2 & 0 & 2 \\ 0 & -4 & 0 \\ 2 & 0 & -2 \end{pmatrix} = \hbar^2\begin{pmatrix} 1/2 & 0 & -1/2 \\ 0 & 1 & 0 \\ -1/2 & 0 & 1/2 \end{pmatrix}$$

$$\hbar^2\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$L_z^2 = \hbar^2\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}^2 = \hbar^2\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$L_x^2 + L_y^2 + L_z^2 = \hbar^2\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \quad \text{OK}$$

stav $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$ v bázi

$L\,0$
$l\,0$
$L\,1$
$l\,1$
$L\,0$
1
$L\,-1$
l_1

$$\mathbb{L}^2 = \hbar^2\begin{pmatrix} 0 & & & \\ & 2 & & 0 \\ & & 2 & \\ 0 & & & 2 \end{pmatrix} \qquad \mathbb{L}_z = \hbar\begin{pmatrix} 0 & & & \\ & 1 & & 0 \\ & & 0 & \\ 0 & & & -1 \end{pmatrix}$$

$$v^T \mathbb{L}^2 v = \frac{1}{2}(1\;0\;1\;0)\,\hbar^2\begin{pmatrix} 0 & & & \\ & 2 & & \\ & & 2 & \\ & & & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \hbar^2$$

$L_z = 0$