Norm, OG

- Gauss $(\;, \langle x\rangle, \langle x^2\rangle)$ kolmé
- vl. stav LHO, vydl: 1st exc, α, bod obratu
- variační? DŮ - uncertainty rel?
- FT, $\langle x\rangle$ ia porep?

ÚKOL

next $a|1\rangle=|0\rangle$ $a|2\rangle=\sqrt{2}|1\rangle$ x^2, matices, π^4, …

ÚKM - TG. 1

Consider $\psi_0(x) = \frac{1}{\sqrt{\alpha\sqrt{\pi}}} e^{-\frac{x^2}{2\alpha^2}}$ & $\psi_1(x) = \sqrt{\frac{2}{\alpha\sqrt{\pi}}}\frac{x}{\alpha} e^{-\frac{x^2}{2\alpha^2}}$

Normaliz: $\int_{-\infty}^{\infty} \psi_0^*(x)\,\psi_0(x)\,dx = \frac{1}{\alpha\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-\frac{x^2}{\alpha^2}} dx$

$\int_{-\infty}^{\infty} e^{-Ax^2} dx = \sqrt{\frac{\pi}{A}}$

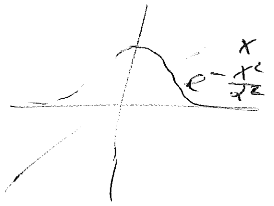
$= \frac{1}{\alpha\sqrt{\pi}} \cdot \alpha\cdot\sqrt{\pi} = 1$ OK

$\int_{-\infty}^{\infty} x^2 e^{-Ax^2} = \frac{1}{2A}\sqrt{\frac{\pi}{A}}$

$\int_{-\infty}^{\infty} \psi_1^*(x)\,\psi_1(x)\,dx = \frac{2}{\alpha\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{x^2}{\alpha^2} e^{-\frac{x^2}{\alpha^2}} dx = \frac{2}{\alpha\sqrt{\pi}}\frac{1}{\alpha^2}\frac{\alpha^2}{2}\alpha\sqrt{\pi} = 1$ OK

$A = \frac{1}{\alpha^2} \to \frac{\alpha^2}{2}\alpha\sqrt{\pi}$

OG: $\int_{-\infty}^{\infty} \psi_0^*(x)\,\psi_1(x)\,dx = \frac{\sqrt{2}}{\alpha\sqrt{\pi}} \underbrace{\int_{-\infty}^{\infty} \frac{x}{\alpha} e^{-\frac{x^2}{\alpha^2}} dx}_{\text{odd / sever}} = 0$

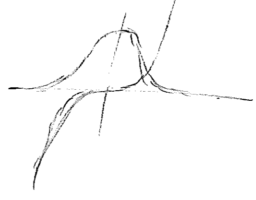


ψ_0:

$\langle x\rangle = \int_{-\infty}^{\infty} \psi_0^*(x)\, x\, \psi_0(x)\,dx = \int_{-\infty}^{\infty} \frac{1}{\alpha\sqrt{\pi}} \underbrace{x\, e^{-\frac{x^2}{\alpha^2}}}_{O/E} dx = 0$

$\langle x^2\rangle = \int_{-\infty}^{\infty} \psi_0^*(x)\, x^2\, \psi_0(x)\,dx = \int_{-\infty}^{\infty} \frac{1}{\alpha\sqrt{\pi}} x^2 e^{-\frac{x^2}{\alpha^2}} dx = \frac{1}{\alpha\sqrt{\pi}}\frac{\alpha^2}{2}\alpha\sqrt{\pi} = \frac{\alpha^2}{2}$

ψ_1: $\langle x\rangle = \int_{-\infty}^{\infty} \psi_1^*(x)\, x\, \psi_1(x)\,dx = \frac{2}{\alpha\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{x^3}{\alpha^2} e^{-\frac{x^2}{\alpha^2}} dx = 0$



$\langle x^2\rangle = \int_{-\infty}^{\infty} \psi_1^*(x)\, x^2\, \psi_1(x)\,dx = \frac{2}{\alpha\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{x^4}{\alpha^2} e^{-\frac{x^2}{\alpha^2}} dx =$

$= \frac{2}{\alpha\sqrt{\pi}\,\alpha^2}\frac{3\alpha^2}{2}\frac{\alpha^2}{2}\alpha\sqrt{\pi} = \frac{3\alpha^2}{2}$

$\int_{-\infty}^{\infty} x^4 e^{-Ax^2} dx = \frac{3}{2A}\frac{1}{2A}\sqrt{\frac{\pi}{A}}$

$A = \frac{1}{\alpha^2}$ $\frac{3\alpha^2}{2}\frac{\alpha^2}{2}\alpha\sqrt{\pi}$

ψ_0: $\langle p^2\rangle = \int_{-\infty}^{\infty} \psi_0^*(x)\, p^2\, \psi_0(x)\,dx = -\hbar^2 \frac{1}{\alpha\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-\frac{x^2}{2\alpha^2}} \frac{d^2}{dx^2} e^{-\frac{x^2}{2\alpha^2}} dx$

$-\hbar^2\frac{d^2}{dx^2}$

$= -\frac{\hbar^2}{\alpha\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-\frac{x^2}{2\alpha^2}} \frac{d}{dx}\left(-\frac{2x}{2\alpha^2} e^{-\frac{x^2}{2\alpha^2}}\right) dx =$

$= -\frac{\hbar^2}{\alpha\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-\frac{x^2}{2\alpha^2}} \left(-\frac{1}{\alpha^2} + \frac{x^2}{\alpha^4}\right) e^{-\frac{x^2}{2\alpha^2}} dx = -\frac{\hbar^2}{\alpha\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-\frac{x^2}{\alpha^2}} \left(-\frac{1}{\alpha^2} + \frac{x^2}{\alpha^4}\right) dx$

$= -\frac{\hbar^2}{\cancel{\alpha\sqrt{\pi}}}\left[-\frac{1}{\alpha^2}\cancel{\alpha\sqrt{\pi}} + \frac{1}{\alpha^4}\frac{\alpha^2}{2}\cancel{\alpha\sqrt{\pi}}\right] = -\hbar^2\left[-\frac{1}{\alpha^2} + \frac{1}{2\alpha^2}\right] = \frac{\hbar^2}{2\alpha^2}$

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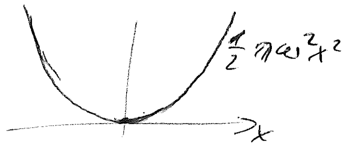
$$\hat{H} = \frac{p^2}{2m} + \frac{1}{2} m \omega^2 x^2$$



$$p = -i\hbar \frac{d}{dx}$$
$$p^2 = -\hbar^2 \frac{d^2}{dx^2}$$
$$\frac{p^2}{2m} = -\frac{\hbar^2}{2m} \frac{d^2}{dx^2}$$

$$\hat{H} \psi_0 = E_0 \psi_0$$

$$\frac{p^2}{2m}\psi_0 = -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} \frac{1}{\sqrt{\alpha\sqrt{\pi}}} e^{-\frac{x^2}{2\alpha^2}}$$

$$= -\frac{\hbar^2}{2m} \frac{1}{\sqrt{\alpha\sqrt{\pi}}} \frac{d}{dx}\left(-\frac{2x}{2\alpha^2} e^{-\frac{x^2}{2\alpha^2}}\right) =$$

$$= +\frac{\hbar^2}{2m} \frac{1}{\sqrt{\alpha\sqrt{\pi}}} \frac{1}{\alpha^2}\left(e^{-\frac{x^2}{2\alpha^2}} - \frac{2x^2}{2\alpha^2} e^{-\frac{x^2}{2\alpha^2}}\right) = \frac{\hbar^2}{2m} \cancel{\hbar} \frac{1}{\alpha^2}\left(1 - \frac{x^2}{\alpha^2}\right) \frac{1}{\sqrt{\alpha\sqrt{\pi}}} e^{-\frac{x^2}{2\alpha^2}}$$

$$\frac{1}{2} m\omega^2 x^2 \psi_0 = \frac{1}{2} m \omega^2 x^2 \frac{1}{\sqrt{\alpha\sqrt{\pi}}} e^{-\frac{x^2}{2\alpha^2}}$$

$$\left(\frac{p^2}{2m} + \frac{1}{2} m\omega^2 x^2\right)\psi_0 = \frac{\hbar^2}{2m\alpha^2}\left(1 - \frac{x^2}{\alpha^2}\right) \cancel{\hbar}\, \psi_0(x) + \frac{1}{2} m \omega^2 x^2 \psi_0(x)$$

$$= \frac{\hbar^2}{2m\alpha^2} \psi_0(x) + \left[-\frac{\hbar^2 x^2}{2m\alpha^4} + \frac{1}{2} m \omega^2 x^2\right] \psi_0(x)$$

no dep. of on x
$\Rightarrow H\psi = E\psi$

dep on $x^2 \to$ no eigenfc.,
only if [ ] = 0 !

$$\to \frac{\hbar^2}{2m\alpha^4} = \frac{1}{2} m\omega^2$$

$$\frac{\hbar^2}{m^2\omega^2} = \alpha^4 \to \alpha = \sqrt{\frac{\hbar}{m\omega}}$$

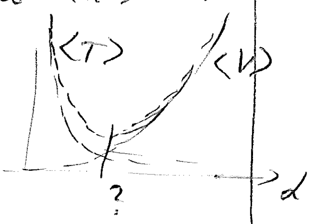
$$\to \left(\frac{p^2}{2m} + \frac{1}{2} m\omega^2 x^2\right)\psi_0 = \frac{\hbar^2}{2m\alpha^2}\psi_0 = \frac{\hbar^2 m\omega}{2m\hbar}\psi_0 = \frac{\hbar\omega}{2}\psi_0$$

E_0

for ψ_0:

$$\langle E\rangle(\alpha) = \left\langle \frac{p^2}{2m} + \frac{1}{2} m\omega^2 x^2\right\rangle(\alpha) = \frac{1}{2m}\langle p^2\rangle(\alpha) + \frac{1}{2} m\omega^2 \langle x^2\rangle(\alpha)$$

$$= \frac{1}{2m}\frac{\hbar^2}{2\alpha^2} + \frac{1}{2} m\omega^2 \frac{\alpha^2}{2} = \frac{\hbar^2}{4m\alpha^2} + \frac{m\omega^2\alpha^2}{4}$$



$$\frac{d\langle E\rangle(\alpha)}{d\alpha} = \frac{d}{d\alpha}\left(\frac{\hbar^2}{4m\alpha^2} + \frac{m\omega^2\alpha^2}{4}\right) = -2\frac{\hbar^2}{4m\alpha^3} + \frac{m\omega^2}{4} 2\alpha$$

$$= -\frac{\hbar^2}{2m\alpha^3} + \frac{m\omega^2\alpha}{2} = 0$$

$$\hookrightarrow \alpha^4 = \frac{\hbar^2}{m^2\omega^2} \to \alpha = \sqrt{\frac{\hbar}{m\omega}}$$

OK

$$\langle T\rangle = \frac{1}{2m}\frac{\hbar^2}{2\alpha^2} = \frac{1\hbar^2}{4m\alpha^2} \stackrel{\alpha}{=} \frac{\hbar^2}{4m}\frac{m\omega}{\hbar} = \frac{\hbar\omega}{4}$$

$$\langle V\rangle = \frac{1}{2} m\omega^2 \frac{\alpha^2}{2} = \frac{m\omega^2\alpha^2}{4} \stackrel{\alpha}{=} \frac{m\omega^2}{4}\frac{\hbar}{m\omega} = \frac{\hbar\omega}{4}$$

ok, so $\hat{H}$ with $V=\frac{1}{2}m\omega^2x^2$ has ψ_0 as solution

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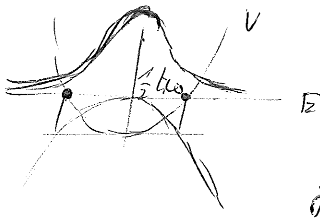

$T(x)$? $\quad T+V=E$

$T=E-V$

for which x: $(T=0)$ $\quad T=\frac{1}{2}\hbar\omega-\frac{1}{2}m\omega^2x^2$

$E=V$

$\frac{1}{2}\hbar\omega=\frac{1}{2}m\omega^2x^2$

$\hbar=m\omega x^2$

$x^2=\frac{\hbar}{m\omega} \Rightarrow x=\sqrt{\frac{\hbar}{m\omega}}=\alpha$

$[x,\frac{d}{dx}]$

$[x\frac{d}{dx}-\frac{d}{dx}x]$

$x\frac{d}{dx}f=xf'$

$\frac{d}{dx}xf=f+xf' \Rightarrow [x,\frac{d}{dx}]=-1$

$f=\frac{1}{\sqrt{\alpha\sqrt{\pi}}}e^{-\frac{x^2}{2\alpha^2}}$

$x\frac{d}{dx}f=x\frac{1}{\sqrt{\alpha\sqrt{\pi}}}\left(-\frac{2x}{2\alpha^2}\right)e^{-\frac{x^2}{2\alpha^2}}=\underline{\underline{-\frac{x^2}{\alpha^2}\frac{1}{\sqrt{\alpha\sqrt{\pi}}}e^{-\frac{x^2}{2\alpha^2}}}}$

$\frac{d}{dx}xf=\frac{d}{dx}x\frac{1}{\sqrt{\alpha\sqrt{\pi}}}e^{-\frac{x^2}{2\alpha^2}}=\frac{1}{\sqrt{\alpha\sqrt{\pi}}}e^{-\frac{x^2}{2\alpha^2}}-x\left(\frac{2x}{2\alpha^2}\right)e^{-\frac{x^2}{2\alpha^2}}=$

$=\frac{1}{\sqrt{\alpha\sqrt{\pi}}}e^{-\frac{x^2}{2\alpha^2}}\underline{\underline{-\frac{x^2}{\alpha^2}e^{-\frac{x^2}{2\alpha^2}}}}$

$[x\frac{d}{dx}-\frac{d}{dx}x]f=\underline{\underline{\quad}}-\frac{1}{\sqrt{\alpha\sqrt{\pi}}}e^{-\frac{x^2}{2\alpha^2}}+\underline{\underline{\quad}}=-\frac{1}{\sqrt{\alpha\sqrt{\pi}}}e^{-\frac{x^2}{2\alpha^2}}\overset{ok}{=}-f$

$\psi_1(x)=\sqrt{\frac{2}{\alpha\sqrt{\pi}}}\frac{x}{\alpha}e^{-\frac{x^2}{2\alpha^2}}$ $\quad ; \frac{p^2}{2m}=-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}$

$\frac{\hbar^2 d^2}{m\,dx^2}\sqrt{\frac{2}{\alpha\sqrt{\pi}}}\frac{x}{\alpha}e^{-\frac{x^2}{2\alpha^2}}=-\frac{\hbar^2}{2m}\sqrt{\frac{2}{\alpha\sqrt{\pi}}}\frac{d}{dx}\left[\frac{1}{\alpha}e^{-\frac{x^2}{2\alpha^2}}-\frac{x}{\alpha}\left(\frac{2x}{2\alpha^2}\right)e^{-\frac{x^2}{2\alpha^2}}\right]=$

$=-\frac{\hbar^2}{2m}\sqrt{\frac{2}{\alpha\sqrt{\pi}}}\frac{1}{\alpha}\left[-\frac{2x}{2\alpha^2}e^{-\frac{x^2}{2\alpha^2}}-\frac{2x}{\alpha^2}e^{-\frac{x^2}{2\alpha^2}}-\frac{x^2}{\alpha^2}\left(-\frac{2x}{2\alpha^2}e^{-\frac{x^2}{2\alpha^2}}\right)\right]=$

$=-\frac{\hbar^2}{2m}\left(\sqrt{\frac{2}{\alpha\sqrt{\pi}}}\frac{1}{\alpha}\right)\left[-\frac{3x}{\alpha^2}e^{-\frac{x^2}{2\alpha^2}}+\frac{x^3}{\alpha^4}e^{-\frac{x^2}{2\alpha^2}}\right]$ $\quad \alpha=\sqrt{\frac{\hbar}{m\omega}}$

$=-\frac{\hbar^2}{2m}\cancel{\sqrt{\frac{2}{\alpha\sqrt{\pi}}}\sqrt{\frac{m\omega}{\hbar}}}\left[=-\frac{\hbar^2}{2m}\psi_0(x)\left[-\frac{3}{\alpha^2}+\frac{x^2}{\alpha^4}\right]=-\frac{\hbar^2}{2m}\psi_0(x)\left[-\frac{3m\omega}{\hbar}+\frac{x^2\,m^2\omega^2}{\hbar^2}\right]\right.$

$=+\hbar\omega\psi_0\left[\frac{3}{2}-\frac{x^2 m\omega}{2\hbar}\right]=\frac{3}{2}\hbar\omega\psi_0\underbrace{-\frac{1}{2}m\omega^2x^2\psi_0}_{=-V\psi_0\ \leftarrow\ \text{cancels}}$

$\frac{1}{2}m\omega^2x^2\psi_0$ ±