Příkl: LHO II

Ökan 2025 T

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$x = \frac{\alpha}{\sqrt{2}}(a^+ + a)$, $p = \frac{i\hbar}{\alpha\sqrt{2}}(a^+ - a)$ $\quad aa^+ = a^+a + 1$

$\langle n|x|n\rangle = \langle n|\frac{\alpha}{\sqrt{2}}(a^+ + a)|n\rangle = \frac{\alpha}{\sqrt{2}}\left[\langle n|\sqrt{n+1}|n+1\rangle + \langle n|\sqrt{n}|n-1\rangle\right]$

1.1 $= 0$

$\langle n|p|n\rangle = 0$

$p^2 = \left[\frac{i\hbar}{\alpha\sqrt{2}}(a^+ - a)\right]^2 = -\frac{\hbar^2}{2\alpha^2}(a^+ - a)(a^+ - a)$

$= -\frac{\hbar^2}{2\alpha^2}(a^{+2} + a^2 - aa^+ - a^+a) = -\frac{\hbar^2}{2\alpha^2}(a^{+2} + a^2 - 2a^+a - 1)$

$\langle n| -\frac{\hbar^2}{2\alpha^2}(a^{+2} + a^2 - 2a^+a - 1)|n\rangle = \frac{\hbar^2}{2\alpha^2}\langle n|2a^+a + 1|n\rangle$

$= \frac{\hbar^2}{2\alpha^2}[2n + 1]$

1.2!

$x^2 = \frac{\alpha^2}{2}[a^+ + a]^2 = \frac{\alpha^2}{2}[a^+a + aa^+ + a^{+2} + a^2] =$

$= \frac{\alpha^2}{2}[a^{+2} + a^2 + 2a^+a + 1]$

$\langle n|x^2|n\rangle = \frac{\alpha^2}{2}[0 + 0 + 2n + 1]$

1.2. $p^2|n\rangle = -\frac{\hbar^2}{2\alpha^2}(a^{+2} + a^2 - 2a^+a - 1)|n\rangle$

$= -\frac{\hbar^2}{2\alpha^2}\left[a^{+2}|n\rangle + a^2|n\rangle - 2a^+a|n\rangle - |n\rangle\right]$

$= -\frac{\hbar^2}{2\alpha^2}\left[\sqrt{n+1}\sqrt{n+2}|n+2\rangle + \sqrt{n}\sqrt{n-1}|n-2\rangle - 2n|n\rangle - |n\rangle\right]$

$\langle n|p^2|n\rangle = -\frac{\hbar^2}{2\alpha^2}\left[0 + 0 - 2n\langle n|n\rangle - \langle n|n\rangle\right]$

$= \frac{\hbar^2}{2\alpha^2}[2n + 1]$

1.3

$H = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2 = -\frac{\hbar^2}{4\alpha^2 m}(a^{+2} + a^2 - 2a^+a - 1) + \frac{1}{2}m\omega^2\frac{\alpha^2}{2}[a^{+2} + a^2 + 2a^+a + 1]$

$= -\frac{\hbar^2}{4m}\frac{m\omega}{\hbar}(a^{+2} + a^2 - 2a^+a - 1) + \frac{1}{4}m\omega^2\frac{\hbar}{m\omega}[a^{+2} + a^2 + 2a^+a + 1]$

$= -\frac{\hbar\omega}{4}(a^{+2} + a^2 - 2a^+a - 1) + \frac{1}{4}\hbar\omega[a^{+2} + a^2 + 2a^+a + 1] = \frac{\hbar\omega}{2}(2a^+a + 1)$

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2.1

$$\langle m|a|n\rangle = \langle m|\sqrt{n}|n-1\rangle = \sqrt{n}\,\delta_{m,n-1}$$

$$\langle m|a^+|n\rangle = \langle m|\sqrt{n+1}|n+1\rangle = \sqrt{n+1}\,\delta_{m,n+1}$$

$$a = \begin{array}{c|c} & n:\ 0 \quad 1 \quad 2 \quad 3 \\ \hline m:\ \begin{matrix}0\\1\\2\end{matrix} & \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & \sqrt2 & 0 \\ 0 & 0 & 0 & \sqrt3 \\ & & & & \ddots \end{pmatrix}\end{array} \qquad a^+ = \begin{array}{c|c} & n:\ 0 \quad 1 \quad 2 \\ \hline m:\ \begin{matrix}0\\1\\2\\3\end{matrix} & \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & \sqrt2 & 0 \\ 0 & 0 & \sqrt3 \\ & & & \ddots \end{pmatrix}\end{array}$$

2.2.

$$aa^+ - a^+a = 1$$

$$aa^+ = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & \sqrt2 & 0 \\ 0 & 0 & 0 & \sqrt3 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & \sqrt2 & 0 \\ 0 & 0 & \sqrt3 \end{pmatrix} = \begin{pmatrix} 1 & & 0 \\ & 2 & \\ 0 & & 3 \end{pmatrix}$$

$$a^+a = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & \sqrt2 & 0 \\ 0 & 0 & \sqrt3 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & \sqrt2 & 0 \\ 0 & 0 & 0 & \sqrt3 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

$$aa^+ - a^+a = \begin{pmatrix} 1 & & \\ & 2 & \\ & & 3 \end{pmatrix} - \begin{pmatrix} 0 & & \\ & 1 & \\ & & 2 \end{pmatrix} = \begin{pmatrix} 1 & & 0 \\ & 1 & \\ 0 & & 1 \end{pmatrix} = \mathbb{1}$$

2.3

$$x = \frac{d}{\sqrt2}(a^+ + a) = \frac{d}{\sqrt2}\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & \sqrt2 & \\ 0 & \sqrt2 & 0 & \sqrt3 \\ & & \sqrt3 & 0 \end{pmatrix}$$

$$x^2 = \frac{d^2}{2}\begin{pmatrix} 1 & 0 & \sqrt2 & \\ 0 & 3 & 0 & \\ \sqrt2 & 0 & 5 & \\ & \sqrt6 & 0 & \ddots \end{pmatrix}$$

$$p = \frac{i\hbar}{d\sqrt2}(a^+ - a) = \frac{i\hbar}{d\sqrt2}\begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & -\sqrt2 & 0 \\ 0 & \sqrt2 & 0 & -\sqrt3 \\ 0 & 0 & \sqrt3 & 0 \end{pmatrix}$$

$$p^2 = -\frac{\hbar^2}{d^2 2}\begin{pmatrix} -1 & 0 & \sqrt2 & 0 \\ 0 & -3 & 0 & \sqrt6 \\ \sqrt2 & 0 & -5 & 0 \\ 0 & -\sqrt6 & 0 & \ddots \end{pmatrix} = \frac{\hbar^2}{2d^2}\begin{pmatrix} 1 & 0 & -\sqrt2 & 0 \\ 0 & 3 & 0 & -\sqrt6 \\ -\sqrt2 & 0 & 5 & 0 \\ 0 & -\sqrt6 & 0 & 7 \end{pmatrix}$$

2.4

$$H = \frac{p^2}{2m} + \frac12 m\omega^2 x^2$$

$$= \frac{\hbar\omega}{4}\begin{pmatrix} 1 & 0 & -\sqrt2 & 0 \\ 0 & 3 & 0 & -\sqrt6 \\ -\sqrt2 & 0 & 5 & 0 \\ 0 & -\sqrt6 & 0 & 7 \end{pmatrix} + \frac{\hbar\omega}{4}\begin{pmatrix} 1 & 0 & \sqrt2 & 0 \\ 0 & 3 & 0 & \sqrt6 \\ \sqrt2 & 0 & 5 & 0 \\ 0 & \sqrt6 & 0 & 7 \end{pmatrix} = \frac{\hbar\omega}{2}\begin{pmatrix} 1 & & & \\ & 3 & & 0 \\ 0 & & 5 & \\ & & & 7 \end{pmatrix}$$

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$$X^3 = X^2 \cdot X = \frac{\alpha^3}{2\sqrt{2}} \begin{pmatrix} 1 & 0 & \sqrt{2} & 0 \\ 0 & 3 & 0 & \sqrt{6} \\ \sqrt{2} & 0 & 5 & 0 \\ 0 & \sqrt{6} & 0 & 7 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & \sqrt{2} & 0 \\ 0 & \sqrt{2} & 0 & \sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{pmatrix}$$

$$= \frac{\alpha^3}{2\sqrt{2}} \begin{pmatrix} 0 & 3 & 0 & \sqrt{6} & 0 & 0 \\ 3 & 0 & 6\sqrt{2} & 0 & 0 & \\ 0 & 6\sqrt{2} & 0 & 9\sqrt{3} & 0 & 0 \\ \sqrt{6} & 0 & 9\sqrt{3} & 0 & 0 & 0 \\ 0 & & 0 & & 0 & \\ & 0 & & & & \end{pmatrix}$$

3.1

$|n\rangle \ldots \psi_n(x)$, čas $E_n = \hbar\omega(n+1/2)$

$$\psi_n(x,t) = \psi_n(x,0)\, e^{-i\hbar\omega(n+1/2)t/\hbar} = \psi_n(x,0)\, e^{-i\omega t(n+1/2)}$$

$$\langle x\rangle(t) = \int_{-\infty}^{\infty} \psi_n^*(x,0)\, \underbrace{e^{+i\omega t(n+1/2)}\, x\, \psi_n(x,0)\, e^{-i\omega t(n+1/2)}}_{1} dx$$

$$= \int_{-\infty}^{\infty} \psi_n^*(x,0)\, x\, \psi_n(x,0)\, dx = 0$$

3.2. $|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$

$$= \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{\alpha\sqrt{\pi}}} e^{-\frac{x^2}{2\alpha^2}} + \sqrt{\frac{2}{\alpha\sqrt{\pi}}}\,\frac{x}{\alpha}\, e^{-\frac{x^2}{2\alpha^2}}\right) = \frac{1}{\sqrt{2}}\frac{1}{\sqrt{\alpha\sqrt{\pi}}}\left(1+\frac{\sqrt{2}x}{\alpha}\right)e^{-\frac{x^2}{2\alpha^2}}$$

$$\psi(x,t) = \frac{1}{\sqrt{2}}\frac{1}{\sqrt{\alpha\sqrt{\pi}}}\left(e^{-i\omega t/2} + \frac{\sqrt{2}x}{\alpha}e^{-3i\omega t/2}\right)e^{-\frac{x^2}{2\alpha^2}}$$

$$\int \psi^*(x,t)\, x\, \psi(x,t)\, dx = \frac{1}{2}\frac{1}{\alpha\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-\frac{x^2}{2\alpha^2}}\left(e^{i\omega t/2} + \frac{\sqrt{2}x}{\alpha}e^{3i\omega t/2}\right) x \left(e^{-i\omega t/2} + \frac{\sqrt{2}}{\alpha}x e^{-3i\omega t/2}\right) e^{-\frac{x^2}{2\alpha^2}} dx$$

$$= \frac{1}{2}\frac{1}{\alpha\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-\frac{x^2}{\alpha^2}} e^{i\omega t/2} x^2 \frac{\sqrt{2}}{\alpha} e^{-3i\omega t/2} + e^{-\frac{x^2}{\alpha^2}}\frac{\sqrt{2}x^2}{\alpha} e^{3i\omega t/2} e^{-i\omega t/2}\, dx$$

$$= \frac{1}{2}\frac{1}{\alpha\sqrt{\pi}}\left[\int_{-\infty}^{\infty} e^{-\frac{x^2}{\alpha^2}} e^{-i\omega t} x^2 \frac{\sqrt{2}}{\alpha} dx + \int_{-\infty}^{\infty} e^{-\frac{x^2}{\alpha^2}} e^{i\omega t} x^2 \frac{\sqrt{2}}{\alpha} dx\right]$$

$$\frac{1}{2A}\sqrt{\frac{\pi}{A}}$$

$$= \frac{1}{2}\frac{1}{\cancel{\alpha\sqrt{\pi}}}\frac{\sqrt{2}}{\alpha}\left[e^{-i\omega t}\frac{\alpha^2}{2}\alpha\sqrt{\pi} + e^{i\omega t}\frac{\alpha^2}{2}\alpha\sqrt{\pi}\right] =$$

$$= \frac{1}{2}\frac{\sqrt{2}}{\alpha}\frac{\alpha^2}{2}\left[e^{-i\omega t} + e^{i\omega t}\right] = \frac{\sqrt{2}\alpha}{2\sqrt{2}} = \frac{\alpha}{\sqrt{2}}\cos(\omega t)$$

3.2 $|\psi\rangle = \frac{1}{\sqrt2}(|0\rangle + |1\rangle)$

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$|\psi\rangle(t) = \frac{1}{\sqrt2}(|0\rangle e^{-i\omega t/2} + |1\rangle e^{-3i\omega t/2})$ $\quad \frac{\alpha}{\sqrt2}(a+a^+)$ $\quad$ 11.4 − 4

$\langle\psi|x|\psi\rangle(t) = \frac{1}{2}\left[\langle 0| e^{i\omega t/2} + \langle 1| e^{3i\omega t/2}\right] x \left[|0\rangle e^{-i\omega t/2} + |1\rangle e^{-3i\omega t/2}\right]$

$= \frac{1}{2}\frac{\alpha}{\sqrt2}\left[\langle 0| e^{i\omega t/2} a |1\rangle e^{-3i\omega t/2} + \langle 1| e^{3i\omega t/2} a^+ |0\rangle e^{-i\omega t/2}\right]$

$= \frac{1}{2}\frac{\alpha}{\sqrt2}\left[e^{-i\omega t}\langle 0|a|1\rangle + e^{i\omega t}\langle 1|a^+|0\rangle\right]$

$= \frac{1}{2}\frac{\alpha}{\sqrt2}\left[e^{-i\omega t} + e^{i\omega t}\right] = \frac{\alpha}{\sqrt2}\cos(\omega t)$ ✓

3.3 $H = \frac{\hbar\omega}{2}(2a^+a + 1)$

$\langle H\rangle(t) = \frac{1}{2}\left[\langle 0| e^{i\omega t/2} + \langle 1| e^{3i\omega t/2}\right]\frac{\hbar\omega}{2}(2a^+a+1)\left[|0\rangle e^{-i\omega t/2} + |1\rangle e^{-3i\omega t/2}\right]$

$= \frac{1}{4}\hbar\omega\left[\langle 0|2a^+a+1|0\rangle + \langle 1|2a^+a+1|1\rangle\right] =$

$= \frac{1}{4}\hbar\omega[1 + 3] = \hbar\omega$ $\quad$ ok (průměr energií)

4. $\hat H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2 - |e| x E$

$\hat H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2\left(x^2 - \frac{2|e|E}{m\omega^2}x\right)$

$A = 1$

$2AB = 2\frac{|e|E}{m\omega^2}$

$B = \frac{|e|E}{m\omega^2}$

$\hat H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2\left(x - \frac{|e|E}{m\omega^2}\right)^2 - \frac{1}{2}m\omega^2\frac{|e|^2E^2}{m^2\omega^4} \to -\frac{1}{2}\frac{|e|^2E^2}{m\omega^2}$

$\hat H = \begin{pmatrix} \hbar\omega/2 & -|e|E\frac{\alpha}{\sqrt2} \\ -\frac{|e|E\alpha}{\sqrt2} & 3\hbar\omega/2 \end{pmatrix}$ $\quad$ $|H - \mathbb{1}\lambda|_{2\times2} = (\frac{1}{2}\hbar\omega - \lambda)(\frac{3}{2}\hbar\omega - \lambda) - \frac{|e|^2E^2\alpha^2}{2}$

$= \lambda^2 - 2\hbar\omega\lambda + \frac{3}{4}\hbar^2\omega^2 - \frac{|e|^2E^2\alpha^2}{2}$

$D = 4\hbar^2\omega^2 - 4\left(\frac{3}{4}\hbar^2\omega^2 - \frac{|e|^2E^2\alpha^2}{2}\right) = \underbrace{4\hbar^2\omega^2 - 3\hbar^2\omega^2}_{\hbar^2\omega^2} + 2|e|^2E^2\alpha^2$

$\lambda_{1/2} = \frac{2\hbar\omega \pm \sqrt{\hbar^2\omega^2 + 2|e|^2E^2\alpha^2}}{2} = \hbar\omega \pm \frac{1}{2}\sqrt{\hbar^2\omega^2 + \frac{2|e|^2E^2\hbar^2}{m\omega\hbar}} = \hbar\omega \pm \frac{1}{2}\hbar\omega\sqrt{1 + \frac{2|e|^2E^2}{m\omega^3\hbar}}$

$\doteq \hbar\omega \pm \frac{1}{2}\hbar\omega\left(1 + \frac{2|e|^2E^2}{m\omega^3\hbar}\right)$

$= \frac{\hbar\omega}{2} - \frac{|e|^2E^2}{m\omega^2}$