Moment hybnosti
– komutační relace

ÚKM-2023
T 25.4.-1

1.1. $[\hat x, \hat p_x] f(x) = \left(x\left(-i\hbar\frac{d}{dx}\right) - \left(-i\hbar\frac{d}{dx}\right)x\right) f(x)$

$= -i\hbar\left(x\frac{d}{dx} - \frac{d}{dx}x\right)f(x) = -i\hbar[xf' - xf' - f] = i\hbar f(x)$

$[\hat x, \hat p_x] = i\hbar$

1.2. $[\hat x, \hat L_x] = [\hat x, \hat y\hat p_z - \hat z\hat p_y] = 0$

$[\hat x, \hat L_y] = [\hat x, \hat z\hat p_x - \hat x\hat p_z] = [\hat x, \hat z\hat p_x] = \hat x\hat z\hat p_x - \hat z\hat p_x\hat x = \hat z[\hat x, \hat p_x] = i\hbar\hat z$

$[\hat x, \hat L_z] = [\hat x, \hat x\hat p_y - \hat y\hat p_x] = [\hat x, -\hat y\hat p_x] = -\hat y[\hat x, \hat p_x] = -i\hbar\hat y$

1.3 $[L_x, L_y] = [yp_z - zp_y, zp_x - xp_z] = [yp_z, zp_x] + [zp_y, xp_z] =$

$= yp_x[p_z, z] + p_y x[z, p_z] = -i\hbar y p_x + i\hbar p_y x = i\hbar L_z$ ✓

1.4. $[L^2, L_x] = [L_x^2 + L_y^2 + L_z^2, L_x] = [L_y^2, L_x] + [L_z^2, L_x]$

$= L_y L_y L_x - L_x L_y L_y + L_z L_z L_x - L_x L_z L_z$

$= L_y L_y L_x - L_y L_x L_y + L_y L_x L_y - L_x L_y L_y + \dots$

$= L_y[L_y, L_x] + [L_y, L_x]L_y + L_z[L_z, L_x] + [L_z, L_x]L_z$

$= -i\hbar L_y L_z - i\hbar L_z L_y + L_z i\hbar L_y + i\hbar L_y L_z = 0$

2.1. $\langle Y_0^0 | Y_0^0\rangle = \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin\theta \, \frac{1}{4\pi} = \frac{2\pi}{4\pi}\int_0^{\pi} d\theta \sin\theta = \frac{1}{2}[-\cos\theta]_0^{\pi} = -\frac{1}{2}[-1-1] = 1$

$\langle Y_1^0 | Y_1^0\rangle = \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin\theta \, \frac{3}{4\pi}\cos^2\theta = \frac{2\pi}{4\pi}\cdot 3\int_0^{\pi} d\theta \sin\theta\cos^2\theta = \frac{3}{2}$ $\cos\theta = t$, $-\sin\theta\, d\theta = dt$

$= -\frac{3}{2}\int_1^{-1} t^2\, dt = -\frac{3}{2}\left[\frac{t^3}{3}\right]_1^{-1} = -\frac{3}{2}\left[-\frac{1}{3} - \frac{1}{3}\right] = 1$

$\sin\theta(1-\cos^2\theta)$

$\langle Y_1^1 | Y_1^1\rangle = \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin\theta \sqrt{\frac{3}{8\pi}} \sin\theta\, e^{-i\phi} \sqrt{\frac{3}{8\pi}} \sin\theta\, e^{i\phi} = \frac{3}{4}\int_0^{\pi} d\theta \sin^3\theta$

$= -\frac{3}{4}\int_1^{-1} dt\,(1-t^2) = -\frac{3}{4}\left[t - \frac{t^3}{3}\right]_1^{-1} = -\frac{3}{4}\left[-1 + \frac{1}{3} - 1 + \frac{1}{3}\right] = -\frac{3}{4}\left[-\frac{4}{3}\right] = 1$

$\langle Y_0^0 | Y_1^0\rangle = \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin\theta \, \frac{1}{\sqrt{4\pi}} \sqrt{\frac{3}{4\pi}} \cos\theta = \frac{\sqrt{3}}{2}\int_0^{\pi} d\theta \sin\theta\cos\theta = \frac{\sqrt{3}}{2}\int_1^{-1} t\, dt = \frac{\sqrt{3}}{2}\left[\frac{t^2}{2}\right]_1^{-1}$

$= \frac{\sqrt{3}}{2}\left[\frac{1}{2} - \frac{1}{2}\right] = 0$

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3 - matice

- báze vl. stavů $\hat{L}^2$, $\hat{L}_z$

$$L^2|l\,m\rangle = \hbar^2 l(l+1)|l\,m\rangle \qquad L_z|l\,m\rangle = \hbar m|l\,m\rangle$$

$$\langle l'm'|L_z|l\,m\rangle = \langle l'm'|\hbar m|l\,m\rangle = \hbar m\langle l'm'|l\,m\rangle = \hbar m\,\delta_{ll'}\delta_{mm'}$$

→ diag. matice

$$L^2 = \hbar^2\begin{pmatrix} 0 & & \\ & \begin{matrix}2 & 0\\ 0 & 2\end{matrix} & 0 \\ 0 & & \begin{matrix}6 & & 0\\ & 6 & \\ 0 & & 6\;6\end{matrix}\end{pmatrix} \qquad L_z: \hbar\begin{pmatrix} 0 & 0 & \\ 0 & \begin{matrix}-1 & & 0\\ & 0 & \\ 0 & & 1\end{matrix} & 0 \\ 0 & & \begin{matrix}-2 & \\ & -1 \\ & & \ddots\end{matrix}\end{pmatrix}$$

záleží na pořadí báz. fcí
zde: $|1\,{-1}\rangle$, $|1\,0\rangle$, $|1\,1\rangle$

4 - $L_\pm|l\,m\rangle = \hbar\sqrt{l(l+1)-m(m\pm1)}\,|l\,m\pm1\rangle$

$$\langle l'm'|L_+|l\,m\rangle = \langle l'm'|\hbar\sqrt{l(l+1)-m(m+1)}\,|l\,m+1\rangle$$
$$= \hbar\sqrt{l(l+1)-m(m+1)}\,\langle l'm'|l\,m+1\rangle$$
$$= \hbar\sqrt{l(l+1)-m(m+1)}\,\delta_{ll'}\delta_{m',m+1}$$

$1(2) \quad -(-1)(-1+1)$

$$\begin{array}{c|ccc} m' \backslash m & -1 & 0 & 1 \\ \hline -1 & 0 & 0 & 0 \\ 0 & \sqrt2 & 0 & 0 \\ 1 & 0 & \sqrt2 & 0 \end{array}$$

L_+ pro $l=0$: (0)

L_+ pro $l=1$: $\hbar\begin{pmatrix} 0 & 0 & 0 \\ \sqrt2 & 0 & 0 \\ 0 & \sqrt2 & 0\end{pmatrix}$

kontrola $|1\,{-1}\rangle = \begin{pmatrix}1\\0\\0\end{pmatrix}$

$$L_+\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ \sqrt2 & 0 & 0 \\ 0 & \sqrt2 & 0\end{pmatrix}\begin{pmatrix}1\\0\\0\end{pmatrix} = \sqrt2\begin{pmatrix}0\\1\\0\end{pmatrix}$$
$= \sqrt2|1\,0\rangle$ OK

L_+ pro $l=2$

$l(l+1) = 2\cdot3 = 6$

$m=-2$: $6-(-2)(-2+1) = 4$

$m=-1$: $6-(-1)(-1+1) = 6$

$m=0$: $6-(0)(\;\;) = 6$

$m=1$: $6-(1)(1+1) = 4$

$$\to \hbar\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 \\ 0 & \sqrt6 & 0 & 0 & 0 \\ 0 & 0 & \sqrt6 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \end{pmatrix}$$

$L_- = (L_+)^\dagger \qquad L_-: l=1: \hbar\begin{pmatrix} 0 & \sqrt2 & 0 \\ 0 & 0 & \sqrt2 \\ 0 & 0 & 0\end{pmatrix}$

4.3. $m_{max} = l$

$$l(l+1) - m_{max}(m_{max}+1) = l(l+1) - l(l+1) = 0$$
$$l(l+1) - m_{min}(m_{min}-1) = l(l+1) - (+l)(+l+1) = 0$$

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4.4. Pro $l=1$: $L_+ = \hbar\begin{pmatrix}0&0&0\\ \sqrt{2}&0&0\\ 0&\sqrt{2}&0\end{pmatrix}$ $\quad L_- = \hbar\begin{pmatrix}0&\sqrt{2}&0\\ 0&0&\sqrt{2}\\ 0&0&0\end{pmatrix}$

$L_+ = L_x + iL_y$
$L_- = L_x - iL_y$

$\rightarrow L_x = \frac{1}{2}(L_+ + L_-) = \frac{1}{\sqrt{2}}\hbar\begin{pmatrix}0&1&0\\1&0&1\\0&1&0\end{pmatrix}$ $\quad L_z = \hbar\begin{pmatrix}-1&&\\&0&\\&&1\end{pmatrix}$

$L_y = \frac{1}{2i}(L_+ - L_-) = \frac{1}{\sqrt{2}i}\hbar\begin{pmatrix}0&-1&0\\1&0&-1\\0&1&0\end{pmatrix}$

4.5
$L_z^2 = \hbar^2\begin{pmatrix}-1&&\\&0&\\&&1\end{pmatrix}\begin{pmatrix}-1&&\\&0&\\&&1\end{pmatrix} = \hbar^2\begin{pmatrix}1&&\\&0&\\&&1\end{pmatrix}$

$L_x^2 = \frac{1}{2}\hbar^2\begin{pmatrix}0&1&0\\1&0&1\\0&1&0\end{pmatrix}\begin{pmatrix}0&1&0\\1&0&1\\0&1&0\end{pmatrix} = \frac{1}{2}\hbar^2\begin{pmatrix}1&0&1\\0&2&0\\1&0&1\end{pmatrix}$ $\quad L_y^2 = -\frac{\hbar^2}{2}\begin{pmatrix}0&-1&0\\1&0&-1\\0&1&0\end{pmatrix}^2 = -\frac{\hbar^2}{2}\begin{pmatrix}-1&0&1\\0&-2&0\\1&0&-1\end{pmatrix} = \frac{\hbar^2}{2}\begin{pmatrix}1&0&-1\\0&2&0\\-1&0&1\end{pmatrix}$

$L_x^2 + L_y^2 + L_z^2 = \hbar^2\left[\begin{pmatrix}1&&\\&2&\\&&1\end{pmatrix} + \begin{pmatrix}1&&\\&0&\\&&1\end{pmatrix}\right] = 2\hbar^2\begin{pmatrix}1&&0\\&1&\\&&1\end{pmatrix} = L^2$

4.6
$[L_x, L_y] = L_xL_y - L_yL_x = \frac{1}{2i}\hbar^2\left[\begin{pmatrix}0&1&0\\1&0&1\\0&1&0\end{pmatrix}\begin{pmatrix}0&-1&0\\1&0&-1\\0&1&0\end{pmatrix} - \begin{pmatrix}0&-1&0\\1&0&-1\\0&1&0\end{pmatrix}\begin{pmatrix}0&1&0\\1&0&1\\0&1&0\end{pmatrix}\right]$

$= \frac{\hbar^2}{2i}\left[\begin{pmatrix}1&0&-1\\0&0&0\\1&0&-1\end{pmatrix} - \begin{pmatrix}-1&0&-1\\0&0&0\\1&0&1\end{pmatrix}\right] = \frac{\hbar^2}{2i}\left[\begin{pmatrix}2&0&0\\0&0&0\\0&0&-2\end{pmatrix}\right] =$

$= i\hbar^2\begin{pmatrix}-1&&0\\0&0&\\&&1\end{pmatrix} = \underline{\underline{i\hbar L_z}}$ OK