Moment hybnosti

1) dokažte, že $\vec{L} = \vec{r} \times \vec{p}$ je hermitovský

např. x: $L_x = y \cdot p_z - z p_y = -i\hbar \left(y \frac{d}{dz} - z \frac{d}{dy} \right)$

$$\langle \psi | L_x \varphi \rangle = \langle L_x \psi | \varphi \rangle$$

$$\int dV\, \psi^* \left[-i\hbar \left(y \frac{\partial}{\partial z} - z \frac{\partial}{\partial y} \right) \right] \varphi \stackrel{?}{=} \int dV \left[i\hbar \left(y \frac{\partial}{\partial z} - z \frac{\partial}{\partial y} \right) \psi^* \right] \varphi$$

$$\int dV\, \psi^* \left[\left(-y \frac{\partial}{\partial z} + z \frac{\partial}{\partial y} \right) \varphi \right] \stackrel{?}{=} \int dV \left[y \frac{\partial \psi^*}{\partial z} - z \frac{\partial \psi^*}{\partial y} \right] \varphi$$

$$\int dV\, \psi^* z \frac{\partial \varphi}{\partial y} + z \frac{\partial \psi^*}{\partial y} \varphi \stackrel{?}{=} \int dV\, y \frac{\partial \psi^*}{\partial z} \varphi + y\, \psi^* \frac{\partial \varphi}{\partial z}$$

$$\int dV\, z \frac{\partial}{\partial y} [\psi^* \varphi] \stackrel{?}{=} \int dV\, y \frac{\partial}{\partial z} [\psi^* \varphi]$$

$0 = 0$ ok

$dx\,dy\,dz$

$$\int_{lim-}^{lim+} \frac{\partial}{\partial y} f(y)\, dy = [f(y)]_{lim-}^{lim+}$$

$$\int z \frac{\partial}{\partial y} [\psi^* \varphi]\, dx\,dy\,dz = \int z [\psi^* \varphi]_{-\infty}^{\infty}\, dx\,dz = 0$$

$\lim_{x \to \pm\infty} \varphi = 0$

2) dokažte, že $\vec{L} \times \vec{L} = i\hbar \vec{L}$ pro $L = r \times p$

bez použití: $[L_i, L_j] = i\hbar \varepsilon_{ijk} L_k$

$\vec{L} \times \vec{L} \to$ výsledek vektor, x-ová složka

$L_y = z p_x - x p_z$

$L_z = x p_y - y p_x$

$$[\vec{L} \times \vec{L}]_x = \cancel{L_y L_z - L_z L_y} = L_y L_z - L_z L_y =$$

$$= (z p_x - x p_z)(x p_y - y p_x) - (x p_y - y p_x)(z p_x - x p_z) =$$

$$= z p_x x p_y - z p_x y p_x - x^2 p_y p_z + x p_x y p_z - x p_x p_y z + x^2 p_y p_z + p_x^2 y z - p_x x y p_z$$

$$= +[p_x, x] p_y z + [x, p_x] y p_z = -i\hbar\, p_y z + i\hbar\, y p_z = i\hbar [y p_z - p_y z] = i\hbar L_x$$

$[x, p_x]$:

$$-i\hbar \left[x, \frac{d}{dx}\right] \psi = -i\hbar \left[x \frac{d\psi}{dx} - \frac{d}{dx}[x\psi] \right] = -i\hbar [-\psi] = i\hbar \psi$$

zbytek cyklicky

3) $L^2 = L_x^2 + L_y^2 + L_z^2$ komutuje s libovolnou složkou $\vec{L}$

$[L^2, L_x] = 0$

$L^2 L_x - L_x L^2 = (\cancel{L_x^2} + L_y^2 + L_z^2) L_x - L_x (\cancel{L_x^2} + L_y^2 + L_z^2) =$

$= L_y^2 L_x + L_z^2 L_x - L_x L_y^2 - L_x L_z^2 =$

$[L_x, L_y] = i\hbar L_z$

$[L_x, L_z] = -i\hbar L_y$

$= L_y (L_y L_x) + L_z (L_z L_x) - (L_x L_y) L_y - (L_x L_z) L_z$

$= L_y [L_y L_x - L_x L_y] + L_z [L_z L_x - L_x L_z]$

$- [L_x L_y - L_y L_x] L_y - [L_x L_z - L_z L_x] L_z$

$= -i\hbar L_y L_z + i\hbar L_z L_y - i\hbar L_z L_y + i\hbar L_y L_z = 0$ ok

$= \cancel{i\hbar [L_y L_z - \ldots}$

4) urči komutační relace složek $\vec{L}$ a x

$[L_x, x] = [y p_z - z p_y, x] = 0$

$[L_y, x] = [z p_x - \cancel{x p_z}, x] = z [p_x, x] = -i\hbar z$

$[L_z, x] = [\cancel{x p_y} - y p_x, x] = -y [p_x, x] = i\hbar y$

5) $\vec{L}$ a p_x

$[L_x, p_x] = [y p_z - z p_y, p_x] = 0$

$[L_y, p_x] = [\cancel{z p_x} - x p_z, p_x] = -[x, p_x] p_z = -i\hbar p_z = -\hbar^2 \frac{\partial}{\partial z}$

$[L_z, p_x] = [x p_y - \cancel{y p_x}, p_x] = [x, p_x] p_y = i\hbar(-i\hbar \frac{\partial}{\partial y}) = \hbar^2 \frac{\partial}{\partial y}$

6) L_x, L_y, L_z ve sférických souřadnicích

$$L_x = -i\hbar\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right) \qquad \Theta = \arccos\left(\frac{z}{r}\right)$$

$$L_y = -i\hbar\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right) \qquad r = (x^2+y^2+z^2)^{1/2}$$

$$\varphi = \operatorname{arctg}\left(\frac{y}{x}\right)$$

$$L_z = -i\hbar\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)$$

$x = r\sin\Theta\cos\varphi$

$y = r\sin\Theta\sin\varphi$

$z = r\cos\Theta$

$$\frac{\partial}{\partial x_\alpha} = \frac{\partial r}{\partial x_\alpha}\frac{\partial}{\partial r} + \frac{\partial\Theta}{\partial x_\alpha}\frac{\partial}{\partial\Theta} + \frac{\partial\varphi}{\partial x_\alpha}\frac{\partial}{\partial\varphi}$$

$$\frac{\partial r}{\partial x_\alpha} = \frac{\partial(x^2+y^2+z^2)^{1/2}}{\partial x_\alpha} = \frac{1}{2}\frac{1}{r}2x_\alpha = \frac{x_\alpha}{r}$$

$$\frac{\partial \arccos(y)}{\partial y} = -\frac{1}{\sqrt{1-y^2}}$$

$$\frac{\partial\Theta}{\partial x} = \frac{\partial\arccos\left(\frac{z}{r}\right)}{\partial x} = -\frac{1}{\sqrt{1-\frac{z^2}{r^2}}}\frac{\partial}{\partial x}\left(\frac{z}{r}\right) = +\frac{1}{\sqrt{1-\frac{z^2}{r^2}}}\left(+\frac{z}{r^2}\right)\frac{x}{r}$$

$$= \frac{1}{\sqrt{x^2+y^2}}\frac{zx}{r^2} = \frac{1}{\cancel{r}\sin\Theta}\frac{\cancel{r}\cos\Theta\, r\sin\Theta\cos\varphi}{r^{\cancel{2}}} = \frac{\cos\Theta\cos\varphi}{r}$$

$$\frac{\partial\varphi}{\partial x} = \frac{\partial\operatorname{arctg}\left(\frac{y}{x}\right)}{\partial x} = \frac{1}{1+\frac{y^2}{x^2}}\left(-\frac{\cancel{y}}{x^2}\right) = -\frac{\cancel{y}}{x^2+y^2} =$$

$$\frac{\partial \operatorname{arctg}(y)}{\partial y} = \frac{1}{1+x^2}$$

$$= -\frac{r\sin\Theta\sin\varphi}{r^2\sin^2\Theta} = -\frac{\sin\varphi}{r\sin\Theta}$$

⋮

$L_z = \ldots$

T 2.3

$$[L_x, r^2] = [y p_z - z p_y, \cancel{x^2} + y^2 + z^2] =$$

$$= [-z p_y, y^2] + [y p_z, z^2] =$$

$$= -z[p_y y^2 - y^2 p_y] + y[p_z z^2 - z^2 p_z]$$

$$= -z[p_y y^2 - y p_y y \quad - y^2 p_y + y p_y y]$$

$$+ y[p_z z^2 - z p_z z \quad - z^2 p_z + z p_z z]$$

$$= -z[p_y, y]y - zy[p_y, y] + y[p_z, z]z + yz[p_z, z]$$

$$= +z i\hbar y + zy i\hbar + yz i\hbar + yz i\hbar = 0$$

$$[L_x, p^2] = [y p_z - z p_y, \cancel{p_x^2} + p_y^2 + p_z^2] =$$

$$= [y p_z, p_y^2] - [z p_y, p_z^2] =$$

$$= p_z[y p_y^2 - p_y^2 y] - p_y[z p_z^2 - p_z^2 z] =$$

$$= p_z[y p_y p_y - p_y y p_y - p_y^2 y + p_y y p_y]$$

$$- p_y[z p_z p_z - p_z z p_z - p_z p_z z + p_z z p_z] =$$

$$= p_z[y, p_y]p_y - p_z p_y[p_y, y] - p_y[z, p_z]p_z + p_y p_z[p_z, z]$$

$$= p_z i\hbar p_y + p_z p_y i\hbar - p_y p_z i\hbar - p_y p_z i\hbar = 0 \text{ OK}$$