Angular momentum   3027 3036 3042   3003 3017 3020   T7N1
Lin 3004   3007 3008   3011 3019 3034

$\hat{L}^2|\psi\rangle = l(l+1)\hbar^2|\psi\rangle \quad \hat{L}_z|\psi\rangle = m\hbar|\psi\rangle$

$[L_i, L_j] = i\hbar\varepsilon_{ijk}L_k$

calculate $\langle L_x\rangle$, $\langle L_x^2\rangle$

$\langle L_x\rangle = \langle\psi|\frac{1}{i\hbar}[L_y, L_z]|\psi\rangle = \frac{1}{i\hbar}\langle\psi|L_yL_z - L_zL_y|\psi\rangle$

$= \frac{m\hbar}{i\hbar}[\langle\psi|L_y|\psi\rangle - \langle\psi|L_y|\psi\rangle] = 0$ ✓ OK

$\langle L_x^2\rangle \underset{\text{symmetry}}{=} \langle L_y^2\rangle = \frac{1}{2}\langle L_x^2 + L_y^2\rangle = \frac{1}{2}\langle L^2 - L_z^2\rangle$

$\rightarrow \langle L_x^2\rangle = \frac{1}{2}\langle\psi|L^2 - L_z^2|\psi\rangle = \frac{1}{2}[l(l+1)\hbar^2 - m^2\hbar^2]$ ✓ OK

$J_+ = J_x + iJ_y \qquad J_\pm|j,m\rangle = \hbar\sqrt{j(j+1) - m(m\pm1)}\,|j, m\pm1\rangle$

$J_- = J_x - iJ_y$

$J_x = \frac{1}{2}(J_+ + J_-)$

$iJ_y = \frac{1}{2}(J_+ - J_-)$

$J_y = \frac{1}{2i}(J_+ - J_-)$

$\cancel{\langle J_x\rangle} = \langle L_x\rangle = \frac{1}{2}\langle L_+ + L_-\rangle = \frac{1}{2}\langle L^+\rangle + \frac{1}{2}\langle L^-\rangle = 0$ ✓ OK

$\langle L_x^2\rangle = \frac{1}{4}\langle\psi|(J_+ + J_-)(J_+ + J_-)|\psi\rangle = \frac{1}{4}\langle\psi|\underbrace{J_+J_+}_{=0} + \underbrace{J_-J_-}_{=0} + J_+J_- + J_-J_+|\psi\rangle$

$= \frac{1}{4}\langle\psi|J_+J_-|\psi\rangle + \frac{1}{4}\langle\psi|J_-J_+|\psi\rangle \qquad |\psi\rangle = |l,m\rangle$

$= \frac{1}{4}\langle l,m|J_+\hbar\sqrt{l(l+1) - m(m-1)}\,|l,m-1\rangle + \frac{1}{4}\langle l,m|J_-\hbar\sqrt{l(l+1) - m(m+1)}\,|l,m+1\rangle$

$= \frac{\hbar^2}{4}\langle l,m|l,m\rangle\sqrt{l(l+1) - m(m-1)}\sqrt{l(l+1) - (m-1)(m-1+1)}$

$+ \frac{\hbar^2}{4}\langle l,m|l,m\rangle\sqrt{l(l+1) - m(m+1)}\sqrt{l(l+1) - (m+1)(m+1-1)}$

$= \frac{\hbar^2}{4}[2l(l+1) - m(m-1) - m(m+1)] = \frac{\hbar^2}{2}[l(l+1) - m^2]$ ✓ OK

using eigenvectors - e.g. for spin

lich 3007 Angular momentum

T7N2

- $s = 1\hbar$, $l = 2\hbar$

$$H_{so} = A\hat{L}\cdot\hat{S}$$ ← find energy levels & degeneracies

J^2, J_z, L^2, S^2 are "good" quantum numbers

$j(j+1)\hbar^2$ $\quad m\hbar$ $\quad l(l+1)\hbar^2$ $\quad s(s+1)\hbar^2$ $\quad \phi_{j m_j l s}$

$\hat{J} = \hat{L} + \hat{S}$

$J^2 = L^2 + S^2 + 2L\cdot S$

$L\cdot S = \frac{1}{2}(J^2 - L^2 - S^2)$

$$H_{so} = A\hat{L}\cdot\hat{S} = \frac{A}{2}(J^2 - L^2 - S^2)$$

$$\rightarrow H_{so}|j j_z l s\rangle = \frac{A}{2}[j(j+1) - \underbrace{l(l+1)}_{=6\hbar^2} - \underbrace{s(s+1)}_{=2\hbar^2}]$$

$= -8\hbar^2$

$l = 2\hbar$, $s = \hbar$

$j = \hbar, 2\hbar, 3\hbar$

$\hbar$	$2\hbar$	$3\hbar$
$\hbar$	$\hbar$	$\vdots$
0	0	
$-\hbar$	$-\hbar$	
	$-2\hbar$	$-3\hbar$

$j = 1\hbar \rightarrow 2\hbar^2 \quad E = -3A\hbar^2$

$j = 2\hbar \quad 6\hbar^2 \quad E = -A\hbar^2$

$j = 3\hbar \quad 12\hbar^2 \quad E = 2A\hbar^2$

$j(j+1)\hbar^2$

3008 $\quad |j, m\rangle = \sum_{l m_l s m_s} c^{jm}_{l m_l s m_s} |l\, m_l; s\, m_s\rangle$

$J_\pm = J_x \pm iJ_y$

$J_\pm |j m\rangle = \hbar\sqrt{j(j+1) - m(m\pm1)}\,|j, m\pm1\rangle$

get $|j, m\rangle$ with $m = l - 1/2$ using $|l, m_l; s\, m_s\rangle$ with $s = 1/2$

$l \otimes 1/2$ A) $j = l + 1/2$; $m = l + 1/2, (l - 1/2), \ldots -l - 1/2$

B) $j = l - 1/2$; $m = (l - 1/2), l - 3/2, \ldots -(l - 1/2)$

A) $j = l + 1/2$

$|l+1/2, l+1/2\rangle = |l, l, 1/2, 1/2\rangle$ ← no other state to contribute

$\quad j \quad m \qquad l \quad m_l \quad s \quad m_s$

$J_-|l+1/2, l+1/2\rangle = \cancel{?\cdot|l+1/2, l-1/2\rangle} = \hbar\sqrt{(l+1/2)(l+1+1/2) - (l+1/2)(l+1/2-1)}\,|l+1/2, l-1/2\rangle$

$J_-|l+1/2, l+1/2\rangle = J_-|l, l; 1/2, 1/2\rangle = (L_- + S_-)|l, l, 1/2, 1/2\rangle$ *

$= \hbar\sqrt{\cancel{l^2} + l + 1/2\, l + 1/2\, l + 1/2 + \cancel{1/4} - \cancel{l^2} - \cancel{l/2} + l - l/2 - \cancel{1/4} + 1/2}\,|l+1/2, l-1/2\rangle$

$= \hbar\sqrt{2l+1}\,|l+1/2, l-1/2\rangle$

* $L_-|l, l; 1/2, 1/2\rangle = \hbar\sqrt{l(l+1) - l(l-1)}\,|l, l-1; 1/2, 1/2\rangle = \hbar\sqrt{2l}\,|l, l-1; 1/2, 1/2\rangle$

$S_-|l, l; 1/2, 1/2\rangle = \hbar\sqrt{1/2(3/2) - 1/2(-1/2)}\,|l, l; 1/2, 1/2\rangle = \hbar\,|l, l; 1/2, -1/2\rangle$

$$|l+1/2, l-1/2\rangle = \sqrt{\frac{2l}{2l+1}}\,|l, l-1; 1/2, 1/2\rangle + \sqrt{\frac{1}{2l+1}}\,|l, l; 1/2, -1/2\rangle$$

$\frac{1}{2}$ spin $\frac{1}{2}$ spin

$\uparrow$ in $\langle S_z\rangle = 1/2$ $\uparrow$ in $\langle S_x\rangle = 1/2$

what's the probability that total spin $= 0$

$\uparrow + \downarrow$ or $\downarrow + \uparrow$

$$|S=0\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\right) \leftarrow \text{in } z \text{ eigenvectors}$$

needs to be antisymmetric

$\frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)$ is $L=1$, $L_z = 0$?

$$S_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \text{ in } S_z \text{ basis}$$

$$\begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 - 1 \qquad \lambda = \pm 1$$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = +1\begin{pmatrix} a \\ b \end{pmatrix} \qquad \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = 0$$

$a = b = \frac{1}{\sqrt{2}}$ for $\frac{\hbar}{2}$

for $-\frac{\hbar}{2}$: $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = 0$ $\qquad a = -b = \frac{1}{\sqrt{2}}$

$$|S_x = 1/2\rangle = \frac{1}{\sqrt{2}}\left(|S_z = 1/2\rangle + |S_z = -1/2\rangle\right)$$

hence

$\overset{\uparrow}{s_z = 1/2}$ $\overset{\rightarrow}{s_x = 1/2}$: $|s_z = \frac{1}{2}, s_x = \frac{1}{2}\rangle = \frac{1}{\sqrt{2}}\left(|s_{z_1} = \frac{1}{2}, s_{z_2} = 1/2\rangle + |s_z = \frac{1}{2}, s_{z_2} = -1/2\rangle\right)$

$$|\langle 0|S_z = 1/2\ S_x = 1/2\rangle|^2 = \frac{1}{4}\Big[\left(\langle s_{z_1} = 1/2, s_{z_2} = -1/2| - \langle s_{z_1} = -1/2, s_{z_2} = 1/2|\right)\cdot$$

$$\cdot\left(|s_{z_1} = 1/2, s_{z_2} = 1/2\rangle + |s_{z_1} = 1/2, s_{z_2} = -1/2\rangle\right)\Big]^2 = \frac{1}{4}$$

$$\left|\begin{matrix} a\uparrow & a\downarrow \\ a\downarrow & a\uparrow \end{matrix}\right| \cancel{\left|\begin{matrix} a\uparrow(1) & a\downarrow(2) \\ a\uparrow(2) & a\downarrow(1) \end{matrix}\right|} =$$

$$\left|\begin{matrix} a\uparrow(1) & a\downarrow(2) \\ a\downarrow(1) & a\uparrow(2) \end{matrix}\right| = a(1)\cdot a(2)\uparrow(1)\uparrow(2) - a(1)\,a(2)\downarrow(1)\downarrow(2)$$

real spin coordinate

$$= \frac{1}{\sqrt{2}} a(1)\,a(2)\left[\uparrow(1)\uparrow(2) - \downarrow(1)\downarrow(2)\right] \quad ?$$

3009 J 7N4

e^- is in state $\psi = \frac{1}{\sqrt{4\pi}} \left(e^{i\phi} \sin\theta + \cos\theta \right) g(r)$
($g(r)$ normalised)

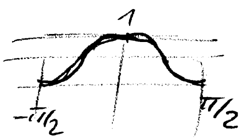
- possible results of measurement of L_z?
- their probability?
- exp. value of L_z?

$Y_{10} = \sqrt{\frac{3}{4\pi}} \cos\theta \; , \; Y_{1,\pm 1} = \mp \sqrt{\frac{3}{8\pi}} \sin\theta \, e^{\pm i\phi}$

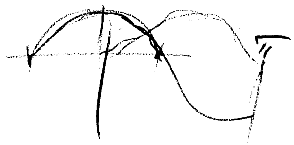
$\langle Y_{10} | \psi \rangle = \int_0^{\pi} d\theta \int_0^{2\pi} d\phi \sqrt{\frac{3}{4\pi}} \frac{1}{\sqrt{4\pi}} \cos\theta \left(e^{i\phi} \sin\theta + \cos\theta \right) \sin\theta =$

Jacobi; 2π periodic $\to 0$

$= \frac{\sqrt{3}}{4\pi} \int_0^{\pi} \cos^2\theta \sin\theta \, d\theta \int_0^{2\pi} d\phi = \frac{\sqrt{3}}{2} \cdot \left[-1/3 \cos^3\theta \right]_0^{\pi}$



$= \frac{\sqrt{3}}{2} (-1/3) \left[-1 - 1 \right] = \frac{1}{\sqrt{3}}$



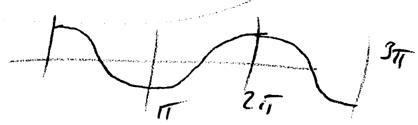
~~$\langle Y_{11} | \psi \rangle = \int_0^{\pi} d\theta \sin\theta \int_0^{2\pi} d\phi \frac{1}{\sqrt{4\pi}} (e^{i\phi}$~~

$\langle Y_{11} | \psi \rangle = \int_0^{\pi} d\theta \int_0^{2\pi} d\phi \left(-\sqrt{\frac{3}{8\pi}} \right) \sin\theta \, e^{-i\phi} \frac{1}{\sqrt{4\pi}} \left(e^{i\phi} \sin\theta + \cos\theta \right) \sin\theta$

$= -\int_0^{\pi} d\theta \int_0^{2\pi} d\phi \sqrt{\frac{3}{2}} \frac{1}{4\pi} \left(\sin^2\theta \cos\theta \, \underset{\uparrow 0}{e^{-i\phi}} + \sin^3\theta \, e^{-i\phi} e^{i\phi} \right) =$

$= -\int_0^{\pi} d\theta \underset{\uparrow 2\pi}{\int_0^{2\pi} d\phi} \sqrt{\frac{3}{2}} \frac{1}{4\pi} \sin^3\theta = -\frac{1}{2} \int_0^{\pi} d\theta \sqrt{\frac{3}{2}} \sin^3\theta =$ $\to \frac{1}{2}$

$\int \sin^3\theta = \frac{1}{12} \cos(3\theta) - \frac{3}{4} \cos\theta$



$= -\frac{1}{2} \sqrt{\frac{3}{2}} \left[\frac{1}{12} \cos(3\theta) - \frac{3}{4} \cos\theta \right]_0^{\pi} = -\frac{1}{2} \sqrt{\frac{3}{2}} \left[\frac{1}{12} (-1-1) - \frac{3}{4} (-1-1) \right]$

$= -\frac{1}{2} \sqrt{\frac{3}{2}} \left[-\frac{1}{6} + \frac{3}{2} \right] = -\frac{1}{2} \sqrt{\frac{3}{2}} \left[-\frac{1}{6} + \frac{9}{6} \right] = -\sqrt{\frac{3}{2}} \frac{4}{3} \frac{1}{2} = -\sqrt{\frac{2}{3}}$

$\to |\psi\rangle = \left(-\sqrt{\frac{2}{3}} |Y_{11}\rangle + \frac{1}{\sqrt{3}} |Y_{10}\rangle \right) g(r)$

$\to L_z$ can be measured as $+\hbar$ or 0.

- Prob. $+\hbar$: $\langle \psi | L_z | \psi \rangle = \frac{2}{3} L_z \hbar + 1/3 L_z 0$

average $= 2/3 \, \hbar$

30 11 T7 N5

operator $f = a + b\vec{\sigma}_1 \cdot \vec{\sigma}_2$ $\frac{1}{2}$ $\frac{1}{2}$ spins, 2 particles

Pauli matrices i.e. ( )( ) + ( )( ) + ( )( )

total spin = $\vec{J} = \vec{j}_1 + \vec{j}_2 = \hbar/2 (\vec{\sigma}_1 + \vec{\sigma}_2)$

$\rho(r) = \sum \phi^*_{s}(r)\phi_{s}(r)$ $e^{im\phi} e^{im\phi}$

a) show that f, J^2, J_z can be simultaneously measured.

•) matrix representation of f in $|J, M, j_1, j_2\rangle$ basis.

•) matrix repr. of f in $|j_1, j_2, m_1, m_2\rangle$ basis.

$\hbar\omega(N + 1/2)$

$[J^2, J_z] = 0$ we know that $\left(J^2 = \frac{3}{2}\hbar I \quad J_z = \frac{\hbar}{2} I \text{ for } s = 1/2 \right)$

$|2 0\rangle$ $|1 1\rangle$ $\hbar^2 l(l+1)$

$J^2 = \hbar^2/4 \, (\sigma_1^2 + \sigma_2^2 + 2\vec{\sigma}_1 \cdot \vec{\sigma}_2)$

$\rightarrow \vec{\sigma}_1 \cdot \vec{\sigma}_2 = \frac{2J^2}{\hbar^2} - \frac{\sigma_1^2}{2} - \frac{\sigma_2^2}{2}$

$\sigma^2 = \sigma_x^2 + \sigma_y^2 + \sigma_z^2 = 3I$ ← identity

$\vec{\sigma}_1 \cdot \vec{\sigma}_2 = \frac{2J^2}{\hbar^2} - 3I$

$|\uparrow\rangle \otimes |\uparrow\rangle$ → $|1 1 \tfrac{1}{2} \tfrac{1}{2}\rangle$

$|\uparrow\rangle \otimes |\downarrow\rangle$ → $|1 0 \tfrac{1}{2} \tfrac{1}{2}\rangle$

$|\downarrow\rangle \otimes |\uparrow\rangle$ → $|1 -1 \tfrac{1}{2} \tfrac{1}{2}\rangle$

$|\downarrow\rangle \otimes |\downarrow\rangle$ → $|0 0 \tfrac{1}{2} \tfrac{1}{2}\rangle$

$[J^2, f] = [J^2, a] + b[J^2, \vec{\sigma}_1 \cdot \vec{\sigma}_2] = 0 + b\left[J^2, \frac{2J^2}{\hbar^2} - 3I\right] = 0$

$[J_z, f] = \underbrace{[J_z, a]}_{0} + b\left[J_z, \underbrace{\frac{2J^2}{\hbar^2}}_{0} - \underbrace{3I}_{0}\right] = 0$ OK

$\langle J, M, j_1, j_2 | f | J', M', j_1, j_2\rangle = \langle J, M, j_1, j_2 | \frac{b 2J^2}{\hbar^2} - 3b + a | J', M', j_1, j_2\rangle$

$= \delta_{JJ'}\delta_{MM'}\left[a - 3b + \frac{2b}{\hbar^2}\hbar^2 J(J+1)\right] = \delta_{JJ'}\delta_{MM'}[a - 3b + 2bJ(J+1)]$

$J = 0 \quad M = 0 \rightarrow a - 3b$

$J = 1 \quad M = 0 \rightarrow a + b$

---

in $|j_1, j_2, m_1, m_2\rangle$ basis

$\psi^{JM}_{00} = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)$ $\quad \psi_{10} = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)$

$\psi_{11} = |\uparrow\uparrow\rangle \quad \psi_{1-1} = |\downarrow\downarrow\rangle$ ← transformation from $m_1 m_2$ basis to LM basis

& transformation the other way round

$|\uparrow\downarrow\rangle = \frac{1}{\sqrt{2}}(\psi_{00} + \psi_{10})$ $\quad |\downarrow\uparrow\rangle = \frac{1}{\sqrt{2}}(\cancel{\psi_{00}} + \psi_{10})$

$= \frac{1}{\sqrt{2}}(-\psi_{00} + \psi_{10})$

$|\uparrow\uparrow\rangle = \psi_{11}$ $\quad |\downarrow\downarrow\rangle = \psi_{1,-1}$

	↑↑	↑↓	↓↑	↓↓
↑↑	a+b	0	0	0
↑↓	0	a−b	2b	0
↓↑	0	2b	a−b	0
↓↓	0	0	0	a+b

OK

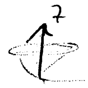
$\langle\uparrow\downarrow| f |\downarrow\uparrow\rangle = 1/2 [-\langle\psi_{00}|f|\psi_{00}\rangle + \langle\psi_{10}|f|\psi_{10}\rangle]$

$= 1/2[-a + 3b + a + b] =$

$= 2b$

3017 T7N6

Φ spin 1/2, measured as $+\hbar/2$

- possible results for measurement along x?
- their probability?
-  for a measure along axis rotated by θ from z, probability of various results?
- expectation value?

---

in σ_z representation $\sigma_x^\uparrow = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$ for $\sigma_x = \hbar/2$ & $\sigma_x^\downarrow = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}$ for $\sigma_x = -\hbar/2$

$s_z = \begin{pmatrix}1\\0\end{pmatrix} = \frac{1}{\sqrt{2}}\left(\sigma_x^\uparrow + \sigma_x^\downarrow\right)$

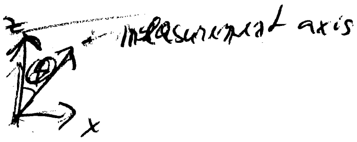
for $+\hbar/2$

$\rightarrow$ can measure both $x\uparrow$ and $x\downarrow$ with 50% probability.

~~$\langle x = 1/2 \mid z = 1/2 \rangle = \langle x = 1/2$~~

$\langle s_x^\uparrow \mid s_z^\uparrow \rangle = \langle s_x^\uparrow \mid \frac{1}{\sqrt{2}}\left(\mid s_x^\uparrow\rangle + \mid s_x^\downarrow\rangle\right)\rangle = \frac{1}{\sqrt{2}}$ prob. $s_x^\uparrow = \left(\frac{1}{\sqrt{2}}\right)^2 = 1/2$

original state   similarly $s_x^\downarrow = 1/2$



$\cos\theta\, \sigma_z + \sin\theta\, \sigma_x$

$\begin{pmatrix}\cos\theta & \sin\theta \\ \sin\theta & -\cos\theta\end{pmatrix}$

~~$\begin{vmatrix}\cos\theta - \lambda & \sin\theta \\ \sin\theta & -\cos\theta - \lambda\end{vmatrix} = (\cos\theta - \lambda)^2 - \sin^2\theta$~~

~~$\lambda_1 = \cos\theta - \sin\theta$~~

~~$\lambda_2 = \cos\theta + \sin\theta$~~

~~$\begin{pmatrix}+\sin\theta & \sin\theta \\ \sin\theta & +\sin\theta\end{pmatrix}\begin{pmatrix} \\ \end{pmatrix} = 0$~~   ~~$\begin{pmatrix}-\sin\theta & \sin\theta \\ \sin\theta & -\sin\theta\end{pmatrix}\begin{pmatrix} \\ \end{pmatrix} = 0$~~

$\begin{vmatrix}\cos\theta - \lambda & \sin\theta \\ \sin\theta & -\cos\theta - \lambda\end{vmatrix} = -(\cos\theta - \lambda)(\cos\theta + \lambda) - \sin^2\theta =$

$= \lambda^2 + \lambda\cos\theta - \lambda\cos\theta - \cos^2\theta - \sin^2\theta = \lambda^2 - 1 \Rightarrow$

$\lambda = \pm 1$

$\begin{pmatrix}\cos\theta - 1 & \sin\theta \\ \sin\theta & -\cos\theta - 1\end{pmatrix}\begin{pmatrix}a\\b\end{pmatrix} = 0$

$\overset{a}{\cos\phi}(\cos\theta - 1) + \overset{b}{\sin\phi}\sin\theta = 0$

$\cos\phi\sin\theta - \sin\phi(\cos\theta + 1) = 0$

$a(\cos\theta - 1) + b\sin\theta = 0$

$a^2 = 1 - b^2$

$a(\cos\theta - 1) + \sqrt{1 - a^2}\sin\theta = 0$

$\cos\phi\cos\theta - \cos\phi + \sin\phi\sin\theta = 0 \quad / \sin\theta$

$\cos\phi\sin\theta - \sin\phi\cos\theta - \sin\phi = 0 \quad / \cos\theta$

$\cos\phi\cos\theta\sin\theta - \cos\phi\sin\theta + \sin\phi\sin^2\theta = 0$

$\cos\phi\cos\theta\sin\theta - \sin\phi\cos\theta - \sin\phi\cos^2\theta = 0$

3017 contd. T7N7

$$-\cos\phi\sin\theta + \sin\phi\cos\theta + \sin\phi(\sin^2\theta + \cos^2\theta) = 0$$

$$-\cos\phi\sin\theta + \sin\phi(\cos\theta + 1) = 0$$

$$\cos\phi\sin\theta - \sin\phi(\cos\theta + 1) = 0$$

$$a(\cos\theta - 1) + b(\sin\theta) = 0$$
$$a\sin\theta - b(\cos\theta - 1) = 0$$

$$a = \cos\theta + \sin\theta = b$$

$$\cos^2\theta - \sin\theta + \sin^2\theta + \cos\theta\sin\theta = 0$$

$$\begin{pmatrix} \cos\theta - 1 & \sin\theta \\ \sin\theta & -\cos\theta - 1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = 0$$

$$a(\cos\theta - 1) + b\sin\theta = 0$$
$$a\sin\theta - b(\cos\theta + 1) = 0$$
$$a(\cos\theta - 1) + \sqrt{1-a^2}\sin\theta = 0$$
$$a^2(\cos\theta - 1)^2 + (1-a^2)\sin^2\theta + 2a\sqrt{1-a^2}(\cos\theta - 1)\sin\theta = 0$$
$$a^2(\cos^2\theta - 2\cos\theta + 1) + \sin^2\theta - a^2\sin^2\theta + 2a\sqrt{1-a^2}(\cos\theta - 1)\sin\theta = 0$$

$$\cos\phi\sin\theta - \sin\phi(\cos\theta + 1) = 0$$
$$\sin\theta - \operatorname{tg}\phi(\cos\theta + 1) = 0$$

$$\boxed{\operatorname{tg}\phi = \frac{\sin\theta}{\cos\theta + 1} \to \therefore \phi = \frac{\theta}{2}}$$ as $\operatorname{tg}\frac{\theta}{2} = \frac{\sin\theta}{\cos\theta + 1}$

$$\begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix} = \cos\theta\,\sigma_z + \sin\theta\,\sigma_x$$

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \begin{pmatrix} 1 \\ 1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

eigenvector is $\begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2} \end{pmatrix}$

for -1: $\begin{pmatrix} \cos\theta + 1 & \sin\theta \\ \sin\theta & -\cos\theta + 1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = 0$

$$a(\cos\theta + 1) + b\sin\theta = 0$$
$$a(\sin\theta) + b(\cos\theta - 1) = 0$$

$$\cos\phi(\cos\theta + 1) + \sin\phi\sin\theta = 0 \to$$
$$\cos\phi(\sin\theta) - \sin\phi(\cos\theta - 1) = 0$$
$$\sin\theta - \operatorname{tg}\phi(\cos\theta - 1) = 0$$
$$\operatorname{tg}\phi = \frac{\sin\theta}{\cos\theta - 1} \qquad \frac{1}{\operatorname{tg}\phi} = \frac{\cos\theta - 1}{\sin\theta}$$

$\begin{pmatrix} -\sin\frac{\theta}{2} \\ \cos\frac{\theta}{2} \end{pmatrix}$

$(\pi/2 - \phi) \to$ cos $\to$ sin, sin $\to$ cos, tg $\to$ ctg

Ang. momentum $\quad J_+ \ \& \ J_-$

$$J_+ = J_x + iJ_y \qquad J_x = \tfrac{1}{2}(J_+ + J_-)$$
$$J_- = J_x - iJ_y$$
$$[J_i, J_j] = i\hbar\, \varepsilon_{\underset{xyz}{ijk}} J_k \qquad J_z J_+ - J_+ J_z = \hbar J_+$$
$$[J_z, J_+] = [J_z, J_x] + i[J_z, J_y] = i\hbar J_y + i i\hbar J_x = \hbar(J_x + iJ_y) = \hbar J_+$$
$$[J_+, J_-] = [J_x + iJ_y, J_x - iJ_y] = i[J_y, J_x] - i[J_x, J_y] = i(-i\hbar J_z) - i(i\hbar J_z) = 2\hbar J_z$$
$$J_z J_+ |l, m\rangle = (\hbar J_+ + J_+ J_z)|lm\rangle = \hbar(1+m) J_+ |lm\rangle$$
$$J_z |lm\rangle = \hbar m |lm\rangle$$
$$\& \ J_z \underbrace{|lm+1\rangle} = \hbar(m+1)|lm+1\rangle$$
$J_+|lm\rangle$ compare $\to J_+ |lm\rangle \to$ must be $\alpha|l\, m+1\rangle$

$$\to J_+ |lm\rangle = \alpha |l\, m+1\rangle$$
$$J_- |lm\rangle = \beta |l\, m-1\rangle$$
$$\langle lm|J_- J_+|lm\rangle = \langle lm|J_+^\dagger J_+|lm\rangle = \langle l m+1|\alpha^* \alpha|l\, m+1\rangle = |\alpha|^2$$
$$\langle lm|J_+ J_-|lm\rangle = \langle lm|J_-^\dagger J_-|lm\rangle = \langle lm-1|\beta^* \beta|lm-1\rangle = |\beta|^2$$
$$J_- J_+ = (J_x - iJ_y)(J_x + iJ_y) = J_x^2 + J_y^2 - iJ_y J_x + iJ_x J_y = J_x^2 + J_y^2 + i[J_x, J_y] =$$
$$= J_x^2 + J_y^2 + i\,i\hbar J_z = J^2 - J_z^2 - \hbar J_z$$
$$J_- J_+ |lm\rangle = J^2 - J_z^2 - \hbar J_z |lm\rangle = \hbar^2 l(l+1) - \hbar^2 m^2 - \hbar^2 m |lm\rangle = \hbar^2[l(l+1) - m(m+1)]|lm\rangle$$
$$\langle lm|J_- J_+|lm\rangle = \hbar^2[l(l+1) - m(m+1)]$$
$$J_+|lm\rangle = \hbar\sqrt{l(l+1) - m(m+1)}\,|l\, m+1\rangle$$
$$J_-|lm\rangle = \hbar\sqrt{l(l+1) - m(m-1)}\,|l\, m-1\rangle \quad \text{in similar spirit}$$

~~so interaction~~

$$\hat{L} = \hat{L}_1 \otimes I_2 + I_1 \otimes \hat{L}_2$$

$$\vec{L}^2 = (\vec{L}_1 \otimes 1_2 + 1_1 \otimes \vec{L}_2)^2 = L_1^2 \otimes 1_2 + 1_1 \otimes L_2^2 + \vec{L}_1 \otimes I_2 \cdot I_1 \otimes \vec{L}_2$$

$$= L_1^2 \otimes 1_2 + 1_1 \otimes L_2^2 + L_1 \otimes L_2$$

$$L_1 \otimes L_2 = \sum_{i=x,y,z} L_{1,i} \otimes L_{2,i}$$

$$[L_{1,z} \otimes 1, L_1 \cdot L_2] = [L_{1,z} \otimes 1, \sum_{i=x,y,z} L_{1,i} \otimes L_{2,i}]$$

$$L_{1,z} \otimes 1, L_{1,x} \otimes L_{2,x} - L_{1,x} \otimes L_{2,x} \cdot L_{1,z} \otimes 1 = L_{1,z} L_{1,x} \otimes L_{2,x} - L_{1,x} L_{1,z} \otimes L_{2x} =$$

$$= i\hbar L_{1,y} \otimes L_{2,x}$$

$$[L_{1,z} \otimes 1, L_{1,y} \otimes L_{2,y}] = -i\hbar L_{1x} \otimes L_{2,y}$$

$\rightarrow L_{1,z}$ doesn't commute with $L_1 \cdot L_2$ but $L_{1,z} + L_{2,z}$ does

$$L_x L_x + \underbrace{L_x L_y + L_x L_z}_{i\hbar L_z} + L_y L_x + L_y L_y + L_y \underbrace{L_z + L_z L_x}_{} + L_z L_y + L_z L_z =$$

$$=$$

$$L^2 = (L_{1x} \otimes 1_2 + 1_1 \otimes L_{2x})^2 + (L_{1y} \otimes 1_2 + 1_1 \otimes L_{2y})^2 + (L_{1z} \otimes 1_2 + 1_1 \otimes L_{2z})^2 =$$

$$= (L_{1x}^2 + L_{1y}^2 + L_{1z}^2) \otimes 1_2 + 1_1 \otimes (L_{2x}^2 + L_{2y}^2 + L_{2z}^2) + 2 L_{1x} \otimes L_{2x} +$$

$$+ 2 L_{1y} \otimes L_{2y} + 2 L_{1z} \otimes L_{2z}$$

$$= 2 \vec{L}_1 \cdot \vec{L}_2$$

$$L_1^2 = L_x L_x + L_y L_y + L_z L_z$$

$$[L_1^2, L_z] = (L_x L_x + L_y L_y) L_z - L_z (L_x L_x + L_y L_y) =$$

$$= L_x L_x L_z + L_y L_y L_z - L_z L_x L_x + L_z L_y L_y =$$

$$= \underbrace{L_x L_x L_z - L_x L_z L_x}_{-i\hbar L_x L_y} + \underbrace{L_x L_z L_x - L_z L_x L_x}_{-i\hbar L_y L_x}$$

$$+ \underbrace{L_y L_y L_z - L_y L_z L_y}_{i\hbar L_y L_x} + \underbrace{L_y L_z L_y - L_z L_y L_y}_{i\hbar L_x L_y} = \emptyset$$

OK

- $L \cdot S$ Hamiltonian $\rightarrow$ trafo to new basis — or $S \cdot S$ Hamiltonian? – double
- addition of ang. momentum, J_+, J_- — a new basis, Hamiltonian? ~~(two spins)~~ $L \cdot S$
- $L \cdot S$ together with S_z

momentum A1 · double ... & "ye olde way"

$A\, S^A \cdot S^B$ Hamiltonian (~~...~~ – in solids $\uparrow\uparrow$ – or magnetic molecules, Ising model, Hard drives.., Kondo effect

~~$S_z \otimes S_z$~~ what does it mean?

original basis:

$$|\tfrac12\ \tfrac12\rangle \otimes |\tfrac12\ \tfrac12\rangle \;\hat{=}\; |\uparrow\uparrow\rangle$$
$$|+\tfrac12\rangle \otimes |-\tfrac12\rangle \qquad |\uparrow\downarrow\rangle$$
$$|-\tfrac12\rangle \otimes |+\tfrac12\rangle \qquad |\downarrow\uparrow\rangle$$
$$|-\tfrac12\rangle \otimes |-\tfrac12\rangle \qquad |\downarrow\downarrow\rangle$$

$A\, S_z^A \otimes S_z^B$

$$A\frac{\hbar}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} \otimes \frac{\hbar}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} = A\frac{\hbar^2}{4}\begin{pmatrix}1&0&0&0\\0&-1&0&0\\0&0&-1&0\\0&0&0&1\end{pmatrix}$$

$\leftarrow$ aligned $\rightarrow$ energy $A\frac{\hbar^2}{4}$ (ferrom.)

antiparallel $\rightarrow$ energy $-A\frac{\hbar^2}{4}$ (antiferrom.)

$\leftarrow$ obviously commutes with S_{zA}, S_{zB}

$$S^A \cdot S^B = S_x^A \otimes S_x^B + S_y^A \otimes S_y^B + S_z^A \otimes S_z^B$$

$$\begin{pmatrix}1\\0\\0\\0\end{pmatrix}\begin{pmatrix}0\\1\\0\\0\end{pmatrix}\begin{pmatrix}0\\0\\1\\0\end{pmatrix}\begin{pmatrix}0\\0\\0\\1\end{pmatrix}$$
basis

$$S_x = \frac{\hbar}{2}\begin{pmatrix}0&1\\1&0\end{pmatrix} \qquad S_y = \frac{\hbar}{2}\begin{pmatrix}0&-i\\i&0\end{pmatrix}$$

$$(S_x^A \otimes S_x^B)^2 = I = (S_y \otimes S_y)^2$$

$$S_x^A \otimes S_x^B = \frac{\hbar^2}{4}\begin{pmatrix}0&0&0&1\\0&0&1&0\\0&1&0&0\\1&0&0&0\end{pmatrix} \qquad S_y^A \otimes S_y^B = \frac{\hbar^2}{4}\begin{pmatrix}0&0&0&-1\\0&0&1&0\\0&1&0&0\\-1&0&0&0\end{pmatrix}$$

$$S_x^A \otimes S_x^B + S_y^A \otimes S_y^B = \frac{\hbar^2}{4}\begin{pmatrix}0&0&0&0\\0&0&2&0\\0&2&0&0\\0&0&0&0\end{pmatrix}$$

$$S^A \cdot S^B = \frac{\hbar^2}{4}\begin{pmatrix}1&0&0&0\\0&-1&2&0\\0&2&-1&0\\0&0&0&1\end{pmatrix}$$

$\leftarrow$ not diagonal in $|\uparrow\uparrow\rangle, |\uparrow\downarrow\rangle, |\downarrow\uparrow\rangle, |\downarrow\downarrow\rangle$ basis

$|\uparrow\uparrow\rangle$ & $|\downarrow\downarrow\rangle$ still ~~go~~ eigenvectors of the new Hamiltonian

$$\begin{pmatrix}-1&2\\2&-1\end{pmatrix} \rightarrow \text{new eigenvectors} \quad \frac{1}{\sqrt2}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle) \qquad E = -1+2 = 1\,\frac{\hbar^2}{4}$$

$$\frac{1}{\sqrt2}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle) \qquad E = -1-2 = -3\,\frac{\hbar^2}{4}$$

$$\begin{pmatrix}0&B\\B&0\end{pmatrix} \rightarrow \begin{pmatrix}-\lambda&B\\B&-\lambda\end{pmatrix} \rightarrow \lambda^2 - B^2 \rightarrow \lambda = B \rightarrow \begin{pmatrix}-B&B\\B&-B\end{pmatrix} \rightarrow \begin{pmatrix}B&B\\B&B\end{pmatrix}$$
$$\searrow \lambda = -B$$

$$\rightarrow \frac{\hbar^2}{4}\begin{pmatrix}1&&&\\&1&&\\&&1&\\&&&-3\end{pmatrix}$$

in $\underbrace{|\uparrow\uparrow\rangle, \tfrac{1}{\sqrt2}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle), |\downarrow\downarrow\rangle}_{\text{triplet}};\ \overbrace{\tfrac{1}{\sqrt2}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)}^{\text{singlet}}$ basis

A $S^A \cdot S^B$ Hamiltonian the other way

$H = A\, S^A \cdot S^B$

$S^A \cdot S^B$: $S^2 = (\vec{S}^A + \vec{S}^B)^2 = S^{A^2} \otimes 1_B + 1_A \otimes S_B^2 + 2\, \underbrace{S^A \cdot S^B}_{S_x^A \otimes S_x^B + S_y^A \otimes S_y^B + S_z^A \otimes S_z^B}$

$S^A \cdot S^B = \frac{1}{2} \cancel{\left(S^2 - S^{A^2} \otimes 1_B + 1_A \otimes S_B^2 \right)}$

$= \frac{1}{2} \left(S^2 - S^{A^2} - S^{B^2} \right)$

two spins with $\frac{\hbar}{2}$ moments $\to$ triplet, singlet

triplet: $S=1$, $S^2 |T\rangle = \hbar^2\, 1(1+1) |T\rangle = 2\hbar^2 |T\rangle$

singlet: $S=0$, $S^2 |S\rangle = \hbar^2\, 0(0+1)|S\rangle = 0\, |S\rangle$

$H|T\rangle = \frac{1}{2}\left(S^2 - S^{A^2} - S^{B^2}\right)|T\rangle =$

$= \frac{1}{2}\hbar^2 \left(\underbrace{1(1+1)}_{2} - \underbrace{\frac{1}{2}\left(\frac{1}{2}+1\right)}_{-\frac{3}{4}} - \underbrace{\frac{1}{2}\left(\frac{1}{2}+1\right)}_{-\frac{3}{4}} \right) |T\rangle$

$= \frac{1}{2}\hbar^2 \frac{1}{2} |T\rangle = \frac{\hbar^2}{4} |T\rangle$ $\leftarrow$ triplet three-fold degenerate

$H|S\rangle = \frac{1}{2}\left(S^2 - S^{A^2} - S^{B^2}\right)|S\rangle =$

$= \frac{1}{2}\hbar^2 \left(0(0+1) - \underbrace{\frac{1}{2}\left(\frac{1}{2}+1\right)}_{-\frac{3}{4}} - \underbrace{\frac{1}{2}\left(\frac{1}{2}+1\right)}_{-\frac{3}{4}} \right) |S\rangle$

$= -\frac{3}{4}\hbar^2 |S\rangle$ $\leftarrow$ singlet - one level

$\to$ energies: $+\frac{\hbar^2}{4}$ $\leftarrow$ triplet; $-\frac{3}{4}\hbar^2$ $\leftarrow$ singlet

we have $A \cdot S^A \cdot S^B$ Hamiltonian
→ diagonal in singlet + triplet basis (coupled)
add $S_z^A \otimes 1^B$ – mag. field in H. atom with L·S coupling – part of (we have S·S)
– Ising model

$$H = S_z^A \otimes 1^B \quad \frac{\hbar}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} \otimes \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix}1 & 0 & & \\ 0 & 1 & & \\ & & -1 & 0\\ & & 0 & -1\end{pmatrix}$$

direct basis diagonal

singlet + triplet basis

$$S_z^A \otimes 1^B |\uparrow\uparrow\rangle = \frac{\hbar}{2}|\uparrow\uparrow\rangle$$

OK

$$S_z^A \otimes 1^B |\downarrow\downarrow\rangle = -\frac{\hbar}{2}|\downarrow\downarrow\rangle$$

triplet states

$$S_z^A \otimes 1^B \frac{1}{\sqrt2}\left(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle\right) = \frac{1}{\sqrt2}\left(\frac{\hbar}{2}|\uparrow\downarrow\rangle - \frac{\hbar}{2}|\downarrow\uparrow\rangle\right) = \frac{\hbar}{2}\cdot|S\rangle$$

$$S_z^A \otimes 1^B \frac{1}{\sqrt2}\left(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\right) = \frac{1}{\sqrt2}\left(\frac{\hbar}{2}|\uparrow\downarrow\rangle + \frac{\hbar}{2}|\downarrow\uparrow\rangle\right) = \frac{\hbar}{2}\cdot|T_0\rangle$$

states $|\uparrow\downarrow\rangle$ and $|\downarrow\uparrow\rangle$ don't have the same eigenvalue
→ their mixture is not an eigenstate

~~in $|T\rangle, |S\rangle$ basis: $\langle \uparrow\downarrow | S_z^A \otimes 1^B \frac{1}{\sqrt2}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle) = \frac{1}{\sqrt2}\frac{\hbar}{2}$~~

H in $|T\rangle, |S\rangle$ basis

$$\begin{pmatrix}\frac{\hbar}{2} & 0 & 0 & \cancel{0}\\ 0 & 0 & \frac{\hbar}{2} & 0\\ \cancel{0} & \frac{\hbar}{2} & 0 & \frac{\hbar}{2}\\ 0 & 0 & 0 & \frac{\hbar}{2}\end{pmatrix}$$

$$\begin{array}{c} \begin{matrix}|T_1\rangle & |T_0\rangle & |T_{-1}\rangle & |S\rangle\end{matrix}\\ \begin{pmatrix}\frac{\hbar}{2} & 0 & 0 & 0\\ 0 & 0 & 0 & \frac{\hbar}{2}\\ 0 & 0 & \frac{\hbar}{2} & 0\\ 0 & \frac{\hbar}{2} & 0 & 0\end{pmatrix}\end{array}$$

+ SS Hamiltonian + S_z

$$\underbrace{A\frac{\hbar^2}{4}}_{A'}\begin{pmatrix}1 & & & \\ & 1 & & \\ & & 1 & \\ & & & -3\end{pmatrix} + \underbrace{B\frac{\hbar}{2}}_{B'}\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 1 & 0\\ 0 & 1 & 0 & 0\end{pmatrix} = \begin{pmatrix}A'+B' & 0 & 0 & 0\\ 0 & A' & 0 & B'\\ 0 & 0 & A'+B' & 0\\ 0 & B' & 0 & -3A'\end{pmatrix}$$

$$\begin{vmatrix}A'-\lambda & B'\\ B' & -3A'-\lambda\end{vmatrix} = (A'-\lambda)(-3A'-\lambda) - B'^2 = -3A'^2 - \lambda A' + 3A'\lambda + \lambda^2 - B'^2 =$$

$$= \lambda^2 + 2A'\lambda - B'^2 - 3A'^2 \qquad D = 4A'^2 + 4B'^2 + 12A'^2 = 16A'^2 + 4B'^2$$

$$\lambda_{1,2} = \frac{-2A' \pm \sqrt{16A'^2 + 4B'^2}}{2} = -A' \pm \sqrt{4A'^2 + B'^2}$$

$B=0$ $\quad \lambda_{1,2} = \begin{matrix}-3A' \\ +A'\end{matrix}$ as before

$B \to 0$

$$-A' \pm 2A'\left(1 + \frac{B'^2}{8A'^2}\right) \Rightarrow \text{splits a bit}$$

$\to$ $S \cdot S$ + S_z Ham:

$$\begin{pmatrix} A'+B' & 0 & 0 & 0 \\ 0 & A' & 0 & B' \\ 0 & 0 & A'+B' & 0 \\ 0 & B' & 0 & -3A' \end{pmatrix}$$

$$\begin{array}{c|cccc} & \uparrow\uparrow & \uparrow\downarrow+\downarrow\uparrow & \downarrow\downarrow & \uparrow\downarrow-\downarrow\uparrow \\ j= & 1 & 1 & 1 & 0 \\ j_z = & 1 & 0 & -1 & 0 \\ S_z^{A,B} & 1/2 & / & -1/2 & / \\ S^2_{A,B} & & & & \end{array}$$

before S_z Ham

Hamiltonian mixes states with different j & same j_z

$S_A^2 \otimes 1_B$
$= 3\, 1_A \otimes 1_B$ $\leftarrow$ still OK

particle is in state with $l = 1\hbar$ and has a spin $1/2$. Momentum-A-4

- What are the possible values of total ang. momentum j & their projections j_z?
- what's j_z using l_z & s_z?
- what's j^2 using $l^2, l_z, l_+, l_-; s^2, s_z, s_-, s_+$

$$j_z = (l_z \otimes 1_s + 1_l \otimes s_z)$$

$$j^2 = (\vec{l} \otimes 1_s + 1_l \otimes \vec{s})^2$$

$$= l^2 \otimes 1_s + 1_l \otimes s^2 + \sum_{xyz} l_i \otimes s_i$$

$$= l^2 \otimes 1_s + 1_l \otimes s^2 + l_x \otimes s_x + l_y \otimes s_y + l_z \otimes s_z$$

$$l_+ = l_x + i l_y \quad \rightarrow \quad l_x = \frac{1}{2}(l_+ + l_-)$$

$$l_- = l_x - i l_y \quad \rightarrow \quad l_y = \frac{1}{2i}(l_+ - l_-)$$

$$l_x \otimes s_x + l_y \otimes s_y = \frac{1}{4}(l_+ + l_-) \otimes (s_+ + s_-) - \frac{1}{4}(l_+ - l_-) \otimes (s_+ - s_-)$$

$$= \frac{1}{4}(2 l_+ \otimes s_- + 2 l_- \otimes s_+) = \frac{1}{2} l_+ \otimes s_- + \frac{1}{2} l_- \otimes s_+$$