ZÁPOČTOVÁ PÍSEMKA 1. 6. 2021 (1)

(1) RUNGEHO-LENZŮV VEKTOR

$$\vec{X} = \frac{1}{2}\left(\vec{L}\times\vec{p} - \vec{p}\times\vec{L}\right) + \vec{n}$$

$$X_i = \frac{1}{2}\left(\varepsilon_{ijk}L_j p_k - \varepsilon_{ijk}p_j L_k\right) + n_i$$

$$H = \frac{p^2}{2} - \frac{1}{r} = -\frac{1}{2}\left(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} - \frac{L^2}{r^2}\right) - \frac{1}{r}$$

I) $A = A^+$ $+$ = hermitovské sdružení
= transpozice + komplexní sdružení

II) $A = A^+,\ B = B^+$: $\underline{(AB)^+} = B^+A^+ = \underline{BA \neq AB}$ ← není obecně herm.

$$\underline{\underline{(AB+BA)^+}} = (AB)^+ + (BA)^+ = B^+A^+ + A^+B^+ = BA + AB = \underline{\underline{AB+BA}}$$

← je herm.

III) $\underline{\underline{[p_i, H]}} = \left[p_i, \frac{p^2}{2} - \frac{1}{r}\right] = \underbrace{\left[p_i, \frac{p^2}{2}\right]}_{=0} - \left[p_i, \frac{1}{r}\right] =$

$= -\left[-i\left(n_i\frac{\partial}{\partial r} + \frac{D_i^n}{r}\right), \frac{1}{r}\right] =$

$= i n_i \left[\frac{\partial}{\partial r}, \frac{1}{r}\right] =$ (ostatní komutují)

$\left[\frac{\partial}{\partial r}, \frac{1}{r}\right] = \frac{\partial}{\partial r}\left(\frac{1}{r}f\right) - \frac{1}{r}\frac{\partial}{\partial r}f = -\frac{1}{r^2}f + \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r}\frac{\partial}{\partial r} = -\frac{1}{r^2}$

$= i n_i\left(-\frac{1}{r^2}\right) = \underline{\underline{-i n_i \frac{1}{r^2}}}$

IV) $\underline{\underline{[L_i, H]}} = \left[\underbrace{-i\varepsilon_{ijk} n_j D_k^n}_{\text{pouze úhl. záv.}}, \frac{p^2}{2} - \frac{1}{r}\right] = \cancel{-i\varepsilon_{ijk}\frac{1}{2}\left[n_j D_k^n, \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r}\right]} \left[L_i, \frac{p^2}{2}\right] =$

$= \left[L_i, -\frac{1}{2}\left(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} - \frac{L^2}{r^2}\right)\right] =$

$= \underbrace{\left[L_i, L^2\right]}_{=0}\frac{1}{2r^2} = \underline{\underline{0}}$

V) $\underline{\underline{[n_i, H]}} = \left[n_i, \frac{p^2}{2} - \frac{1}{r}\right] = \frac{1}{2}\left[n_i, p^2\right] =$

$= \left[n_i, -\frac{1}{2}\left(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} - \frac{L^2}{r^2}\right)\right] =$

$= \frac{1}{2r^2}\left[n_i, L^2\right] = \underline{\underline{\frac{1}{r^2}\left(D_i^{(n)} - n_i\right)}}$

VI) $\underline{[p_j L_k, H]} = p_j [L_k, H] + [p_j, H] L_k =$

$= p_j \cdot 0 + (-i n_j \frac{1}{r^2}) L_k =$

$= \underline{-i \frac{1}{r^2} n_j L_k}$

$[L_j p_k, H] = [L_j, H] p_k + L_j [p_k, H] =$

$= L_j (-i n_k \frac{1}{r^2}) =$

$= -i \frac{1}{r^2} L_j n_k$

VII) $\underline{[X_i, H]} = \left[\frac{1}{2}(\varepsilon_{ijk} L_j p_k - \varepsilon_{ijk} p_j L_k) + n_i , H\right] =$

$= \frac{1}{2} \varepsilon_{ijk} \underbrace{[L_j p_k, H]}_{-i \frac{1}{r^2} L_j n_k} - \frac{1}{2} \varepsilon_{ijk} \underbrace{[p_j L_k, H]}_{-i \frac{1}{r^2} n_j L_k} + \underbrace{[n_i, H]}_{\frac{1}{r^2}(D_i^{(n)} - n_i)} =$

$= -i \frac{1}{2} \varepsilon_{ijk} L_j n_k \frac{1}{r^2} + i \frac{1}{2} \varepsilon_{ijk} n_j L_k \frac{1}{r^2} + \frac{1}{r^2}(D_i^{(n)} - n_i) =$ $\qquad L_i = -i \varepsilon_{ijk} n_j D_k^n$

$= -i \frac{1}{2} \varepsilon_{ijk} (-i) \varepsilon_{jpq} n_p D_q^n n_k \frac{1}{r^2} + i \frac{1}{2} \varepsilon_{ijk} n_j (-i) \varepsilon_{kpq} n_p D_q^{(n)} \frac{1}{r^2} + \frac{1}{r^2}(D_i^n - n_i) =$

$= -\frac{1}{2}(\delta_{kp}\delta_{iq} - \delta_{kq}\delta_{ip}) n_p D_q^{(n)} n_k \frac{1}{r^2} + \frac{1}{2}(\delta_{ip}\delta_{jq} - \delta_{iq}\delta_{jp}) n_j n_p D_q^n \frac{1}{r^2} + \frac{1}{r^2}(D_i^n - n_i) =$

$\cancel{= -\frac{1}{2} n_k D_i^n n_k \frac{1}{r^2} + \frac{1}{2}}$

$= -\frac{1}{2r^2}(\underbrace{n_k D_i^n n_k} - n_i \underbrace{D_k^n n_k}_{2}) + \frac{1}{2r^2}(\underbrace{n_j n_i D_j^n}_{= n_i n_j D_j^n = 0} - \underbrace{n_j n_j}_{=1} D_i^{(n)}) + \frac{1}{r^2}(D_i^n - n_i) =$

$n_k (n_k D_i^n + \delta_{ik} - n_i n_k) =$

$= D_i^n + n_i - n_i$

$= -\frac{1}{2r^2}(D_i^n - 2n_i) + \frac{1}{2r^2}(- D_i^{(n)}) + \frac{1}{r^2}(D_i^n - n_i) =$

$= -\frac{1}{r^2} D_i^n + \frac{1}{r^2} D_i^n + \frac{1}{r^2} n_i - \frac{1}{r^2} n_i = \underline{0}$

② LHO

VIII) $H = \frac{P^2}{2} + \frac{X^2}{2} = a^+a + \frac{1}{2}$ $\quad a = \frac{1}{\sqrt{2}}(x+ip)$ $\quad a|n\rangle = \sqrt{n}\,|n-1\rangle$

$a^+ = \frac{1}{\sqrt{2}}(x-ip)$ $\quad a^+|n\rangle = \sqrt{n+1}\,|n+1\rangle$

$E_n = n + \frac{1}{2}$

IX) $x^3 = \left[\frac{1}{\sqrt{2}}(a+a^+)\right]^3 = \frac{1}{\sqrt{8}}(aa + aa^+ + a^+a + a^+a^+)(a+a^+) =$

$= \frac{1}{\sqrt{8}}(aaa + aa^+a + a^+aa + a^+a^+a +$
$+ aaa^+ + aa^+a^+ + a^+aa^+ + a^+a^+a^+)$

X) $x^3|n\rangle = \frac{1}{\sqrt{8}}\Big(\sqrt{n(n-1)(n-2)}\,|n-3\rangle + \sqrt{n^3}\,|n-1\rangle + \sqrt{n(n-1)^2}\,|n-1\rangle + \sqrt{n^2(n+1)}\,|n+1\rangle +$
$+ \sqrt{(n+1)^2 n}\,|n-1\rangle + \sqrt{(n+1)^2(n+2)^2}\,|n+1\rangle + \sqrt{(n+1)^3}\,|n+1\rangle + \sqrt{(n+1)(n+2)(n+3)}\,|n+3\rangle\Big)$

XI) $E_0^{(1)} = \langle 0|x^3|0\rangle = \underline{\underline{0}}$

$E_1^{(1)} = \langle 1|x^3|1\rangle = \underline{\underline{0}}$

XII) $E_0^{(2)} = \sum_{k\neq 0} \frac{|\langle 0|x^3|k\rangle|^2}{E_0^{(0)} - E_k^{(0)}} = \sum_{k\neq 0} \frac{|\langle 0|x^3|k\rangle|^2}{-k} =$ $\quad k = 1,3$

$E_0^{(0)} = \frac{1}{2}$, $\quad E_k^{(0)} = k + \frac{1}{2}$

$= \frac{|\langle 0|x^3|1\rangle|^2}{-1} + \frac{|\langle 0|x^3|3\rangle|^2}{-3} =$

$= -\left(\frac{3}{\sqrt{8}}\right)^2 - \left(\frac{\frac{\sqrt{6}}{\sqrt{8}}}{3}\right)^2 =$

$= -\frac{9}{8} - \frac{2}{8} = \underline{\underline{-\frac{11}{8}}}$

$\langle 0|x^3|1\rangle = \left[\begin{array}{l}\langle 0|aaa^+|1\rangle = \frac{1}{\sqrt{8}}\cdot 2 = \frac{1}{\sqrt{2}} \\ + \langle 0|aa^+a|1\rangle = \frac{1}{\sqrt{8}}\cdot 1\end{array}\right]$

$\langle 0|x^3|3\rangle = \frac{1}{\sqrt{8}}\langle 0|aaa|3\rangle = \frac{1}{\sqrt{8}}\sqrt{6} = \frac{\sqrt{3}}{2}$

XIII) $|\Psi\rangle = N(|0\rangle + 2|1\rangle - |2\rangle)$

$\langle\Psi|\Psi\rangle = N^2(1+4+1) \doteq 1 \Rightarrow \underline{\underline{N = \frac{1}{\sqrt{6}}}}$

XIV) $P_0 = |\langle 0|\Psi\rangle|^2 = \left|\frac{1}{\sqrt{6}}\right|^2 = \underline{\underline{\frac{1}{6}}}$

$P_1 = |\langle 1|\Psi\rangle|^2 = \left|\frac{2}{\sqrt{6}}\right|^2 = \frac{4}{6} = \underline{\underline{\frac{2}{3}}}$ $\quad$ kontrola $\frac{1}{6} + \frac{2}{3} + \frac{1}{6} = 1$ ✓

$P_2 = |\langle 2|\Psi\rangle|^2 = \left|\frac{1}{\sqrt{6}}\right|^2 = \underline{\underline{\frac{1}{6}}}$

XV) $\langle\Psi|H|\Psi\rangle = \frac{1}{6}\cdot\underbrace{\left(\frac{1}{2}\right)}_{E_0} + \frac{4}{6}\cdot\underbrace{\left(\frac{3}{2}\right)}_{E_1} + \frac{1}{6}\underbrace{\left(\frac{5}{2}\right)}_{E_2} = \frac{1}{12} + 1 + \frac{5}{12} = \frac{1+12+5}{12} = \frac{18}{12} = \underline{\underline{\frac{3}{2}}}$

(3) TROJJAMA

$$H = \tau\left(|1\rangle\langle 2| + |2\rangle\langle 1| + |1\rangle\langle 3| + |3\rangle\langle 1| + |2\rangle\langle 3| + |3\rangle\langle 2|\right)$$

$P|1\rangle = |2\rangle \quad P|2\rangle = |3\rangle \quad P|3\rangle = |1\rangle$

XVI) $H = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$

$P = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$

XVII) $H \cdot P = \tau\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} = \tau\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$

$P \cdot H = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}\tau\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} = \tau\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$

$\Bigg\}\ [H, P] = HP - PH = \underline{\underline{0}}$

XVIII) vl. stavy P:

$$\det\begin{vmatrix} -l_i & 0 & 1 \\ 1 & -l_i & 0 \\ 0 & 1 & -l_i \end{vmatrix} = -l_i^3 + 1 \overset{!}{=} 0$$

$$l_i = \sqrt[3]{1}$$

$-l_i c_1 + c_3 = 0 \quad \Rightarrow \quad c_3 = l_i c_1$

$c_1 - l_i c_2 = 0 \quad \Rightarrow \qquad\qquad \Rightarrow \quad \underline{\underline{|l_i\rangle = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ l_i^2 \\ l_i \end{pmatrix}}}$

$c_2 - l_i c_3 = 0 \quad \Rightarrow \quad c_2 = l_i c_3 = l_i^2 c_1$

$|c_1|^2 + |c_2|^2 + |c_3|^2 \overset{!}{=} 1 \; : \; |c_1|^2 + |l_i^2 c_1|^2 + |l_i c_1|^2 =$

$= |c_1|^2 \underbrace{(1 + |l_i|^2 + |l_i|^2)}_{3} \overset{!}{=} 1 \quad \Rightarrow \quad |c_1| = \frac{1}{\sqrt{3}}$

XIX) vl. stavy H:

$$\det\begin{vmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{vmatrix} = -\lambda^3 - 3\lambda + 2 \overset{!}{=} 0$$

$$-(\lambda - 2)(\lambda + 1)^2 = 0$$

$\lambda_1 = 2 \quad \Rightarrow \quad \underline{E_1 = 2\tau}$ nedeg.

$\lambda_{2,3} = -1 \quad \Rightarrow \quad \underline{\underline{E_2 = -\tau}}$ 2x deg.

XX) H

XX)
$$H|l_i\rangle = \tau\begin{pmatrix}0&1&1\\1&0&1\\1&1&0\end{pmatrix}\frac{1}{\sqrt{3}}\begin{pmatrix}1\\l_i^2\\l_i\end{pmatrix} =$$

$$= \frac{\tau}{\sqrt{3}}\begin{pmatrix}l_i^2+l_i\\1+l_i\\1+l_i^2\end{pmatrix}$$

$l_i = 1 = l_0$: $H|l_0\rangle = \frac{\tau}{\sqrt{3}}\begin{pmatrix}2\\2\\2\end{pmatrix} = 2\tau\frac{1}{\sqrt{3}}\begin{pmatrix}1\\1\\1\end{pmatrix}$ $\quad$ $l_0 = 1$ odpovídá $E = 2\tau$

$l_i \neq 1 \equiv l_1, l_2$: $H|l_i\rangle = \tau\frac{1}{\sqrt{3}}\begin{pmatrix}-1\\-l_i^2\\-l_i\end{pmatrix} = (-1)\tau\frac{1}{\sqrt{3}}\begin{pmatrix}1\\l_i^2\\l_i\end{pmatrix}$ $\quad$ $l_{1,2}$ odpovídá $E = -\tau$

$1 + l_i + l_i^2 = 0$ pro $l_i \neq 1$ $\qquad$ komplexní

XXI)

$l_0 = l_0^2 = 1$ $\qquad |l_0\rangle = \frac{1}{\sqrt{3}}\begin{pmatrix}1\\1\\1\end{pmatrix}$

$l_1 = l_2^2$ $\qquad |l_1\rangle = \frac{1}{\sqrt{3}}\begin{pmatrix}1\\l_2\\l_1\end{pmatrix}$

$l_2 = l_1^2$ $\qquad |l_2\rangle = \frac{1}{\sqrt{3}}\begin{pmatrix}1\\l_1\\l_2\end{pmatrix}$

$$|l_0\rangle + |l_1\rangle + |l_2\rangle = \frac{1}{\sqrt{3}}\begin{pmatrix}1+1+1\\1+l_2+l_1\\1+l_1+l_2\end{pmatrix} = \frac{1}{\sqrt{3}}\begin{pmatrix}3\\0\\0\end{pmatrix} \Rightarrow |1\rangle = \frac{1}{\sqrt{3}}\left(|l_0\rangle + |l_1\rangle + |l_2\rangle\right)$$

$$1 + l_1 + l_2 = 1 + l_1 + l_1^2 = 0$$