Úvodní kvíz

1. • hermitovský operátor $H^+ = H$ $+$ = hermitovské sdružení
= "transpozice + komplexní sdružení"
$$H_{ij} = H_{ji}^*$$

• vlastní čísla jsou reálná
• vlastní vektory jsou navzájem kolmé
(a tvoří úplný systém)

2. unitární operátor $U^{-1} = U^*$
$$\Rightarrow U^*U = UU^* = \mathbb{1}$$

3. spektrum matice = množina vl. čísel matice
$$A.\vec{v} = \lambda\vec{v}$$

4.
$$(1\ 0)\begin{pmatrix}1\\ i\end{pmatrix} = 1.1 + 0.i = \underline{1}$$

$$\begin{pmatrix}1\\ -1\end{pmatrix}(1\ -1) = \begin{pmatrix}1.1 & 1.(-1)\\ (-1).1 & (-1).(-1)\end{pmatrix} = \underline{\underline{\begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix}}}$$

$$\begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix}\begin{pmatrix}1\\ 0\end{pmatrix} = \begin{pmatrix}1.1 + (-1).0\\ (-1).1 + 1.0\end{pmatrix} = \underline{\underline{\begin{pmatrix}1\\ -1\end{pmatrix}}}$$

$$\begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix}\begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix} = \begin{pmatrix}1.1 + (-1).(-1) & 1.(-1) + (-1).1\\ (-1).1 + 1.(-1) & (-1).(-1) + 1.1\end{pmatrix} = \underline{\underline{2\begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix}}}$$

5. vlastní čísla + vlastní vektory

• $\begin{pmatrix}1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & -1\end{pmatrix}$ matice diagonální $\Rightarrow$ $\lambda_1 = 1$ $v_1 = \begin{pmatrix}1\\ 0\\ 0\end{pmatrix}$

$\lambda_2 = 0$ $v_2 = \begin{pmatrix}0\\ 1\\ 0\end{pmatrix}$

$\lambda_3 = -1$ $v_3 = \begin{pmatrix}0\\ 0\\ 1\end{pmatrix}$

• $\begin{pmatrix}0 & 1 & 0\\ 1 & 0 & 1\\ 0 & 1 & 0\end{pmatrix}\begin{pmatrix}a\\ b\\ c\end{pmatrix} = \lambda\begin{pmatrix}a\\ b\\ c\end{pmatrix}$

$\Rightarrow$ řešíme soustavu
$$\left.\begin{aligned} b &= \lambda a\\ a + c &= \lambda b \end{aligned}\right\}$$ ~~stačí~~ jen 2 rce, + podmínka normalizace (nebo parametr)
$b = \lambda c$ ← ~~obsažena~~ takhle už obsažena
$$|a|^2 + |b|^2 + |c|^2 = 1$$

$a = t$ t = parametr
$b = \lambda t$
$c = \lambda b - a = \lambda^2 t - t = (\lambda^2 - 1)t$

vl. čísla:
$$\left[\begin{pmatrix}0 & 1 & 0\\ 1 & 0 & 1\\ 0 & 1 & 0\end{pmatrix} - \lambda\overbrace{\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}}^{\mathbb{1}}\right]\begin{pmatrix}a\\ b\\ c\end{pmatrix} = 0 \Leftrightarrow \det\begin{vmatrix}-\lambda & 1 & 0\\ 1 & -\lambda & 1\\ 0 & 1 & -\lambda\end{vmatrix} \stackrel{!}{=} 0$$

$$\det\begin{vmatrix} -\lambda & 1 & 0 \\ 1 & -\lambda & 1 \\ 0 & 1 & -\lambda \end{vmatrix} = (-\lambda)^3 + 1\cdot1\cdot0 + 0\cdot1\cdot1 - 0\cdot(-\lambda)\cdot0 - (-\lambda)\cdot1\cdot1 - 1\cdot1\cdot(-\lambda) =$$

$$= -\lambda^3 + 2\lambda = \lambda(2-\lambda^2)$$

$\det|\dots| \stackrel{!}{=} 0$: $\lambda(2-\lambda^2) = 0 \Rightarrow \lambda_1 = \sqrt{2}$

$\lambda_2 = 0$

$\lambda_3 = -\sqrt{2}$

a odpovídající vl. vektory:

$\lambda_1 = \sqrt{2}$: $a = t$, $b = \sqrt{2}t$, $c = t$ $\quad v_1 = \begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix}\cdot t, \ t \in \mathbb{R}$ $\quad$ norm.: $t^2 + 2t^2 + t^2 = 1$, $t^2 = \frac{1}{4}$, $|t| = \frac{1}{2}$ $\Rightarrow v_1 = \frac{1}{2}\begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix}$

$\lambda_2 = 0$: $a = t$, $b = 0$, $c = -t$ $\quad v_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} t, \ t \in \mathbb{R}$ $\quad$ norm.: $t^2 + t^2 = 1$, $|t| = \frac{1}{\sqrt{2}}$ $\Rightarrow v_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$

$\lambda_3 = -\sqrt{2}$: $a = t$, $b = -\sqrt{2}t$, $c = t$ $\quad v_3 = \begin{pmatrix} 1 \\ -\sqrt{2} \\ 1 \end{pmatrix} t, \ t \in \mathbb{R}$ $\quad$ norm. $\Rightarrow v_3 = \frac{1}{2}\begin{pmatrix} 1 \\ -\sqrt{2} \\ 1 \end{pmatrix}$

• $\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$: $\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \lambda \begin{pmatrix} a \\ b \end{pmatrix}$

$\det\begin{pmatrix} -\lambda & -i \\ i & -\lambda \end{pmatrix} \stackrel{!}{=} 0$: $\lambda^2 - i\cdot(-i) = \lambda^2 - 1 = 0$

$\lambda_{1,2} = \pm 1$

$\lambda_1 = +1$: $\left.\begin{aligned} -ib &= a \\ ia &= b \\ |a|^2 + |b|^2 &= 1 \end{aligned}\right\} \Rightarrow$ $|-ib|^2 + |b|^2 = 2|b|^2 = 1$, $|b| = \frac{1}{\sqrt{2}}$, $a = \frac{1}{\sqrt{2}}\cdot(-i)$ $\Rightarrow v_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} -i \\ 1 \end{pmatrix}$

$\lambda_2 = -1$: $\left.\begin{aligned} -ib &= -a \\ ia &= -b \\ |a|^2 + |b|^2 &= 1 \end{aligned}\right\} \Rightarrow$ $|-ib|^2 + |b|^2 = 1$, $|b| = \frac{1}{\sqrt{2}}$, $a = \frac{1}{\sqrt{2}}\cdot i$ $\Rightarrow v_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} i \\ 1 \end{pmatrix}$

(6) $\delta_{ij} = 1 \quad i = j$

$\phantom{\delta_{ij}} = 0 \quad i \neq j$

! pozor na sčítání (Einsteinova sumační konvence) $\delta_{ii} = \delta_{11} + \delta_{22} + \delta_{33} = 3$

(7) plně antisym. tenzor

$\varepsilon_{ijk} = 1$ pro sudé permutace $(1,2,3), (2,3,1), (3,1,2)$

$\varepsilon_{ijk} = -1$ pro liché permutace $(1,3,2), (2,1,3), (3,2,1)$

$\varepsilon_{ijk} = 0$ jinak

$\varepsilon_{ijk}\varepsilon_{ilm} = \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}$! pamatovat