1. cvičení (2.3.2021) – úlohy z textu

(I) $\underline{\underline{P_2}} = |\langle 2|\psi\rangle|^2 = \langle 2|\psi\rangle\langle\psi|2\rangle = c_2 \underbrace{\langle 2|2\rangle}_{=1} \cdot c_2^* \underbrace{\langle 2|2\rangle}_{1} = \underline{\underline{|c_2|^2}}$

(II) hermitovsky sdružené vektory:

$|1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \langle 1| = (1 \ \ 0)$

$|-1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad \langle -1| = (0 \ \ 1)$

$|\psi\rangle = c_1|1\rangle + c_2|2\rangle - \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \quad \Rightarrow \quad \langle\psi| = (c_1^* \ \ c_2^*)$

(III) trojhladinový systém:

bázové vektory: $|1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad |2\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad |3\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

operátor $\hat{S}$ v této bázi: $\hat{S} = \begin{pmatrix} \langle 1|\hat{S}|1\rangle & \langle 1|\hat{S}|2\rangle & \langle 1|\hat{S}|3\rangle \\ \langle 2|\hat{S}|1\rangle & \langle 2|\hat{S}|2\rangle & \langle 2|\hat{S}|3\rangle \\ \langle 3|\hat{S}|1\rangle & \langle 3|\hat{S}|2\rangle & \langle 3|\hat{S}|3\rangle \end{pmatrix}$

(IV) $\langle +z|-z\rangle = (1 \ \ 0)\begin{pmatrix} 0 \\ 1 \end{pmatrix} = 1\cdot 0 + 0\cdot 1 = 0$

$\langle -z|+z\rangle = (0 \ \ 1)\begin{pmatrix} 1 \\ 0 \end{pmatrix} = 0$

(V)

$\hat{S}_i = \begin{pmatrix} \langle +x|S_i|+x\rangle & \langle +x|S_i|-x\rangle \\ \langle -x|S_i|+x\rangle & \langle -x|S_i|-x\rangle \end{pmatrix}$

(VI)

$\hat{S}_z|+z\rangle = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{2}|+z\rangle$

$\hat{S}_z|-z\rangle = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = -\frac{1}{2}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = -\frac{1}{2}|-z\rangle$

(VII) $\hat{S}_\pm|l,m\rangle = \sqrt{l(l+1) - m(m\pm 1)}\ |l, m\pm 1\rangle$

$\hat{S}_+|\tfrac{1}{2},+\tfrac{1}{2}\rangle = \sqrt{\tfrac{1}{2}(\tfrac{1}{2}+1) - \tfrac{1}{2}(\tfrac{1}{2}+1)}\ |\tfrac{1}{2},+\tfrac{3}{2}\rangle = \underline{\underline{0}}$

$\underline{\underline{\hat{S}_+|\tfrac{1}{2},-\tfrac{1}{2}\rangle}} = \sqrt{\tfrac{1}{2}(\tfrac{1}{2}+1) - (-\tfrac{1}{2})(-\tfrac{1}{2}+1)}\ |\tfrac{1}{2},+\tfrac{1}{2}\rangle = \sqrt{\tfrac{3}{4}+\tfrac{1}{4}}\ |\tfrac{1}{2},+\tfrac{1}{2}\rangle = \underline{\underline{|\tfrac{1}{2},+\tfrac{1}{2}\rangle}}$

$\underline{\underline{\hat{S}_-|\tfrac{1}{2},+\tfrac{1}{2}\rangle}} = \sqrt{\tfrac{1}{2}(\tfrac{1}{2}+1) - \tfrac{1}{2}(\tfrac{1}{2}-1)}\ |\tfrac{1}{2},-\tfrac{1}{2}\rangle = \underline{\underline{|\tfrac{1}{2},-\tfrac{1}{2}\rangle}}$

$\underline{\underline{S_-|\tfrac{1}{2},-\tfrac{1}{2}\rangle}} = \sqrt{\tfrac{1}{2}(\tfrac{1}{2}+1) - (-\tfrac{1}{2})(-\tfrac{1}{2}-1)}\ |\tfrac{1}{2},-\tfrac{3}{2}\rangle = \underline{\underline{0}}$

(VIII) $\frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \lambda\begin{pmatrix} a \\ b \end{pmatrix}$

$\begin{pmatrix} -\lambda & \frac{1}{2} \\ \frac{1}{2} & -\lambda \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = 0 \quad \Rightarrow \quad \det\begin{pmatrix} -\lambda & \frac{1}{2} \\ \frac{1}{2} & -\lambda \end{pmatrix} = \lambda^2 - \frac{1}{4} \overset{!}{=} 0$

$\underline{\underline{\lambda = \pm\frac{1}{2}}}$

$\lambda_1 = +\frac{1}{2}: \quad \frac{1}{2}b = \frac{1}{2}a \Rightarrow a = b$

$|a|^2 + |b|^2 = 1 \quad \Rightarrow \quad \underline{\underline{v_1}} = \begin{pmatrix} a \\ b \end{pmatrix} = \underline{\underline{\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}}}$

$\lambda_2 = -\frac{1}{2}: \quad \frac{1}{2}b = -\frac{1}{2}a \Rightarrow a = -b$

$|a|^2 + |b|^2 = 1 \quad \Rightarrow \quad \underline{\underline{v_2}} = \begin{pmatrix} a \\ b \end{pmatrix} = \underline{\underline{\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}}}$

IX $\hat{S}_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

$\hat{S}_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

$S_\pm = S_x \pm i S_y \Rightarrow 2 i S_y = S_+ - S_-$

$$\underline{\underline{S_y}} = \frac{1}{2i}(S_+ - S_-) = \frac{1}{2i}\left[\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\right] =$$

$$= \frac{-i}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} =$$

$$= \underline{\underline{\frac{1}{2}\begin{pmatrix} 0 & -i \\ +i & 0 \end{pmatrix}}}$$

vlastní čísla a stavy $\hat{S}_y$ (v bázi $\{|+z\rangle, |-z\rangle\}$):

$\frac{1}{2}\begin{pmatrix} 0 & -i \\ +i & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \lambda \begin{pmatrix} a \\ b \end{pmatrix}$ $\qquad S_y|\pm y\rangle = \lambda|\pm y\rangle$

$\hookrightarrow \det\begin{pmatrix} -\lambda & -\frac{i}{2} \\ +\frac{i}{2} & -\lambda \end{pmatrix} \stackrel{!}{=} 0 : \quad \lambda^2 - \frac{1}{4} = 0$

$$\underline{\underline{\lambda = \pm \frac{1}{2}}}$$

$\lambda_1 = +\frac{1}{2}$: $\quad -\frac{i}{2} b = \frac{1}{2} a \Rightarrow a = -ib$

$|a|^2 + |b|^2 = 1 \qquad \Rightarrow \underline{\underline{v_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix}}}$

$\lambda_2 = -\frac{1}{2}$: $\quad -\frac{i}{2} b = -\frac{1}{2} a \Rightarrow a = ib$

$|a|^2 + |b|^2 = 1 \qquad \Rightarrow \underline{\underline{v_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix}}}$

X $\underline{\underline{P_{+z,-x}}} = |\langle +z|-x\rangle|^2 = \langle +z|-x\rangle\langle -x|+z\rangle =$

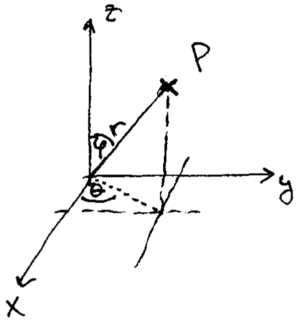
$$= \left|(1\ 0)\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right|^2 =$$

$$= \left|\frac{1}{\sqrt{2}}\right|^2 =$$

$$= \underline{\underline{\frac{1}{2}}}$$

2. CVIČENÍ (9.3.2021)

(I) sférické souřadnice



$$x = r\sin\theta\cos\varphi$$
$$y = r\sin\theta\sin\varphi$$
$$z = r\cos\theta$$
$$\vec r = r\cdot\vec n$$

(II) $\|\vec n\| = \vec n\cdot\vec n = n_x n_x + n_y n_y + n_z n_z =$

$$= \sin\theta\cos\varphi\sin\theta\cos\varphi + \sin\theta\sin\varphi\sin\theta\sin\varphi + \cos\theta\cos\theta =$$
$$= \sin^2\theta\underbrace{(\cos^2\varphi + \sin^2\varphi)}_{=1} + \cos^2\theta =$$
$$= \sin^2\theta + \cos^2\theta =$$
$$= 1$$

(III) $\hat S_n = \vec{\hat S}\cdot\vec n = \hat S_x n_x + \hat S_y n_y + \hat S_z n_z =$

$$= \frac{1}{2}\begin{pmatrix}0&1\\1&0\end{pmatrix}\sin\theta\cos\varphi + \frac{1}{2}\begin{pmatrix}0&-i\\+i&0\end{pmatrix}\sin\theta\sin\varphi + \frac{1}{2}\begin{pmatrix}1&0\\0&-1\end{pmatrix}\cos\theta =$$
$$= \frac{1}{2}\begin{pmatrix}\cos\theta & \sin\theta(\cos\varphi - i\sin\varphi)\\ \sin\theta(\cos\varphi + i\sin\varphi) & -\cos\theta\end{pmatrix} =$$

Eulerův vztah $e^{ix} = \cos x + i\sin x$

$$= \frac{1}{2}\begin{pmatrix}\cos\theta & \sin\theta\, e^{-i\varphi}\\ \sin\theta\, e^{i\varphi} & -\cos\theta\end{pmatrix}$$

(IV) $S_n|n\rangle = \lambda|n\rangle \Rightarrow \det(S_n - \mathbb{1}\cdot\lambda) = 0$

$$\begin{vmatrix}\frac{1}{2}\cos\theta - \lambda & \frac{1}{2}\sin\theta e^{-i\varphi}\\ \frac{1}{2}\sin\theta e^{i\varphi} & -\frac{1}{2}\cos\theta - \lambda\end{vmatrix} = -\frac{1}{4}\cos^2\theta + \lambda^2 - \frac{1}{4}\sin^2\theta = \lambda^2 - \frac{1}{4} = 0$$
$$\Rightarrow \lambda = \pm\frac{1}{2}$$

(V)
$$\cos\theta\, x + \sin\theta\, e^{-i\varphi} y = x \quad (1)$$
$$\sin\theta\, e^{+i\varphi} x - \cos\theta\, y = y \quad (2)$$

(1): $x(\cos\theta - 1) + \sin\theta\, e^{-i\varphi} y = 0$

$$e^{+i\varphi}\sin\theta(\cos\theta - 1)x + \underbrace{\sin^2\theta}_{1-\cos^2\theta} y = 0$$
$$(\cos\theta - 1)\sin\theta\, e^{+i\varphi}x - (\cos\theta - 1)(\cos\theta + 1)y = 0$$
$$\sin\theta\, e^{+i\varphi}x - \cos\theta\, y = y \quad \doteq (2)$$

(VI) $|x|^2 + |y|^2 = 1$

$$N^2\sin^2\theta + N^2(1-\cos\theta)^2 = 2N^2(1-\cos\theta) = 1$$
$$\Rightarrow N = \frac{1}{2\sqrt{1-\cos\theta}} \quad \Rightarrow \quad v_1 = \frac{1}{\sqrt{2(1-\cos\theta)}}\begin{pmatrix}\sin\theta\, e^{-i\varphi}\\ 1-\cos\theta\end{pmatrix} = \begin{pmatrix}\cos\frac{\theta}{2}e^{-i\varphi}\\ \sin\frac{\theta}{2}\end{pmatrix}$$

VIII $[S_y,S_z] = S_yS_z - S_zS_y = \frac{1}{2}\begin{pmatrix} & -i\\ i & \end{pmatrix}\frac{1}{2}\begin{pmatrix}1 & \\ & -1\end{pmatrix} - \frac{1}{2}\begin{pmatrix}1 & \\ & -1\end{pmatrix}\frac{1}{2}\begin{pmatrix} & -i\\ i & \end{pmatrix} =$

$$= \frac{1}{4}\begin{pmatrix}0 & i\\ i & 0\end{pmatrix} - \frac{1}{4}\begin{pmatrix}0 & -i\\ -i & 0\end{pmatrix} =$$

$$= i\,\frac{1}{2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} = iS_x$$

$[S_z,S_x] = S_zS_x - S_xS_z = \frac{1}{2}\begin{pmatrix}1 & \\ & -1\end{pmatrix}\frac{1}{2}\begin{pmatrix} & 1\\ 1 & \end{pmatrix} - \frac{1}{2}\begin{pmatrix} & 1\\ 1 & \end{pmatrix}\frac{1}{2}\begin{pmatrix}1 & \\ & -1\end{pmatrix} =$

$$= \frac{1}{4}\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix} - \frac{1}{4}\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix} =$$

$$= i\,\frac{1}{2}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix} = iS_y$$

VII $[\hat{A}\hat{B},\hat{C}] = \hat{A}\hat{B}\hat{C} - \hat{C}\hat{A}\hat{B}$

$A[B,C] + [A,C]B = ABC - \cancel{ACB} + \cancel{ACB} - CAB = ABC - CAB$ □

IX $[S^2,S_i] = [S_jS_j, S_i] = S_j[S_j,S_i] + [S_j,S_i]S_j =$

$$= S_j\, i\varepsilon_{jik}S_k + i\varepsilon_{jik}S_kS_j =$$

$$= i\underbrace{\varepsilon_{jik}}_{\text{antisym.}}\underbrace{(S_jS_k + S_kS_j)}_{\text{sym.}} = 0$$

8 $[W_1,W_2] = [r, rp_r] = r\underbrace{[r,p_r]}_{i} + \underbrace{[r,r]}_{=0}p_r = ir = iW_1$

$[W_2,W_3] = [rp_r, rp_r^2] = r[p_r, rp_r^2] + [r, rp_r^2]p_r =$

$$= r\underbrace{[p_r,r]}_{-i}p_r^2 + r\underbrace{[r,p_r^2]}p_r =$$

$$p_r\underbrace{[r,p_r]}_{i} + \underbrace{[r,p_r]}_{i}p_r = 2ip_r$$

$$= -irp_r^2 + r\,2ip_rp_r =$$

$$= irp_r^2 = iW_3$$

$[W_1,W_3] = [r, rp_r^2] = r[r,p_r^2] = r\,2ip_r = 2iW_2$