CVIČENÍ ÚDKM ①

(I) $\underline{S_z|\uparrow\downarrow\rangle} = S_{z1}|\uparrow\rangle \otimes \mathbb{I}_2|\downarrow\rangle + \mathbb{I}_1|\uparrow\rangle \otimes S_{z2}|\downarrow\rangle =$

$= \frac{1}{2}|\uparrow\rangle\otimes|\downarrow\rangle + (-\frac{1}{2})|\uparrow\rangle\otimes|\downarrow\rangle =$

$= \underline{0\,|\uparrow\downarrow\rangle}$

$\underline{S_z|\downarrow\uparrow\rangle} = S_{z1}|\downarrow\rangle\otimes\mathbb{I}_2|\uparrow\rangle + \mathbb{I}_1|\downarrow\rangle\otimes S_{z2}|\uparrow\rangle =$

$= -\frac{1}{2}|\downarrow\rangle\otimes|\uparrow\rangle + \frac{1}{2}|\downarrow\rangle\otimes|\uparrow\rangle =$

$= \underline{0\,|\downarrow\uparrow\rangle}$

$\underline{S_z|\downarrow\downarrow\rangle} = S_{z1}|\downarrow\rangle\otimes\mathbb{I}_2|\downarrow\rangle + \mathbb{I}_1|\downarrow\rangle\otimes S_{z2}|\downarrow\rangle =$

$= -\frac{1}{2}|\downarrow\rangle\otimes|\downarrow\rangle + (-\frac{1}{2})|\downarrow\rangle\otimes|\downarrow\rangle =$

$= \underline{-1\,|\downarrow\downarrow\rangle}$

(II) $\hat{\vec{A}}\cdot\hat{\vec{B}} = A_1B_1 + A_2B_2 + A_3B_3$

$= \frac{1}{2}(A_+B_- + A_-B_+) + A_3B_3$

$A_\pm = A_1 \pm iA_2$

$B_\pm = B_1 \pm iB_2$

$\underline{\frac{1}{2}(A_+B_- + A_-B_+) + A_3B_3} = \frac{1}{2}\left[(A_1+iA_2)(B_1-iB_2) + (A_1-iA_2)(B_1+iB_2)\right] + A_3B_3 =$

$= \frac{1}{2}\left[A_1B_1 + iA_2B_1 - iA_1B_2 + A_2B_2 + \right.$

$\left. + A_1B_1 - iA_2B_1 + iA_1B_2 + A_2B_2\right] + A_3B_3 =$

$= \underline{A_1B_1 + A_2B_2 + A_3B_3}$

(III) $\hat{S}^2 = \hat{S}_1^2 + \hat{S}_2^2 + (\hat{S}_{1+}\hat{S}_{2-} + \hat{S}_{1-}\hat{S}_{2+}) + 2\hat{S}_{1z}\hat{S}_{2z}$

$\underline{\hat{S}^2|\uparrow\downarrow\rangle} = \hat{S}_1^2|\uparrow\rangle\otimes\mathbb{I}_2|\downarrow\rangle + \mathbb{I}_1|\uparrow\rangle\otimes\hat{S}_2^2|\downarrow\rangle + \hat{S}_{1+}|\uparrow\rangle\otimes S_{2-}|\downarrow\rangle + S_{1-}|\uparrow\rangle\otimes S_{2+}|\downarrow\rangle + 2S_{1z}|\uparrow\rangle\otimes S_{2z}|\downarrow\rangle$

$= \frac{1}{2}(\frac{1}{2}+1)|\uparrow\rangle|\downarrow\rangle + \frac{1}{2}(\frac{1}{2}+1)|\uparrow\rangle|\downarrow\rangle + 0 + |\downarrow\rangle|\uparrow\rangle + 2(+\frac{1}{2})(-\frac{1}{2})|\uparrow\rangle|\downarrow\rangle =$

$= (\frac{3}{4} + \frac{3}{4} - \frac{1}{2})|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle =$

$= \underline{|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle}$

$$\boxed{\begin{array}{ll} S_+|\uparrow\rangle = 0 & S_+|\downarrow\rangle = |\uparrow\rangle \\ S_-|\downarrow\rangle = 0 & S_-|\uparrow\rangle = |\downarrow\rangle \\ S_z|\uparrow\rangle = \frac{1}{2}|\uparrow\rangle & S_z|\downarrow\rangle = -\frac{1}{2}|\downarrow\rangle \\ \multicolumn{2}{l}{S^2|\uparrow\rangle = \frac{1}{2}(\frac{1}{2}+1)|\uparrow\rangle = \frac{3}{4}|\uparrow\rangle} \\ \multicolumn{2}{l}{S^2|\downarrow\rangle = \frac{1}{2}(\frac{1}{2}+1)|\downarrow\rangle = \frac{3}{4}|\downarrow\rangle} \end{array}}$$

$\underline{S^2|\downarrow\uparrow\rangle} = S_1^2|\downarrow\rangle\otimes\mathbb{I}_2|\uparrow\rangle + \mathbb{I}_1|\downarrow\rangle\otimes S_2^2|\uparrow\rangle +$

$+ S_{1+}|\downarrow\rangle\otimes S_{2-}|\uparrow\rangle + S_{1-}|\downarrow\rangle\otimes S_{2+}|\uparrow\rangle +$

$+ 2S_{z1}|\downarrow\rangle\otimes S_{z2}|\uparrow\rangle =$

$= \frac{3}{4}|\downarrow\rangle|\uparrow\rangle + \frac{3}{4}|\downarrow\rangle|\uparrow\rangle + |\uparrow\rangle|\downarrow\rangle + 0 + 2\cdot(-\frac{1}{2})\cdot\frac{1}{2}|\downarrow\rangle|\uparrow\rangle =$

$= (\frac{3}{4} + \frac{3}{4} - \frac{1}{2})|\downarrow\uparrow\rangle + |\uparrow\downarrow\rangle = \underline{|\downarrow\uparrow\rangle + |\uparrow\downarrow\rangle}$

$$\underline{\underline{S^2|\downarrow\downarrow\rangle}} = S_1^2|\downarrow\rangle \otimes \mathbb{I}_2|\downarrow\rangle + \mathbb{I}_1|\downarrow\rangle \otimes S_2^2|\downarrow\rangle + S_{1+}|\downarrow\rangle \otimes S_{2-}|\downarrow\rangle + S_{1-}|\downarrow\rangle \otimes S_{2+}|\downarrow\rangle + 2\, S_{z1}|\downarrow\rangle \otimes S_{z2}|\downarrow\rangle =$$

$$= \frac{3}{4}|\downarrow\rangle|\downarrow\rangle + \frac{3}{4}|\downarrow\rangle|\downarrow\rangle + 0 + 0 + 2\cdot\left(-\frac{1}{2}\right)\cdot\left(-\frac{1}{2}\right)|\downarrow\rangle|\downarrow\rangle =$$

$$= \left(\frac{3}{4}+\frac{3}{4}+\frac{1}{2}\right)|\downarrow\downarrow\rangle =$$

$$= \underline{\underline{2\,|\downarrow\downarrow\rangle}}$$

IV skládáme $j_1 = \frac{1}{2}$ a $j_2 = \frac{1}{2}$

$$|\uparrow\rangle = |\tfrac{1}{2}, \tfrac{1}{2}\rangle \equiv |+\rangle$$
$$|\downarrow\rangle = |\tfrac{1}{2}, -\tfrac{1}{2}\rangle \equiv |-\rangle$$

$$|j_1, m_{j_1}\rangle$$

1. najdeme možná j: $j = |j_1 - j_2|, \dots, j_1 + j_2$

$$\underline{\underline{j = 0, 1}}$$

2. možné stavy: → abychom získali výsledné m, $m = m_1 + m_2$

m	$\|m_1\rangle\|m_2\rangle$		$\|j,m\rangle$
$+1$	$\|+\rangle\|+\rangle$	$\|\uparrow\uparrow\rangle$	$\|1,1\rangle$
0	$\|+\rangle\|-\rangle$, $\|-\rangle\|+\rangle$	$\|\uparrow\downarrow\rangle$, $\|\downarrow\uparrow\rangle$	$\|1,0\rangle$, $\|0,0\rangle$
-1	$\|-\rangle\|-\rangle$	$\|\downarrow\downarrow\rangle$	$\|1,-1\rangle$

3. triviální kombinace

$$|1,1\rangle = |+\rangle|+\rangle = |\uparrow\uparrow\rangle$$
$$|1,-1\rangle = |-\rangle|-\rangle = |\downarrow\downarrow\rangle$$

4. zbývající kombinace → pro $m=0$

↳ určíme možné hodnoty i

$$|j_1\, i\rangle\, |j_2\, m-i\rangle$$

$$\underset{\equiv|+\rangle}{|\tfrac{1}{2}, \tfrac{1}{2}\rangle}\, \underset{\equiv|-\rangle}{|\tfrac{1}{2}, 0-\tfrac{1}{2}\rangle} \quad \text{a} \quad \underset{\equiv|-\rangle}{|\tfrac{1}{2}, -\tfrac{1}{2}\rangle}\, \underset{\equiv|+\rangle}{|\tfrac{1}{2}, 0-(-\tfrac{1}{2})\rangle} \quad \Rightarrow \text{tj. } i \in \{-\tfrac{1}{2}, \tfrac{1}{2}\}$$

5. vyřešíme rovnici $\qquad \alpha^\pm(j,m) = \sqrt{j(j+1) - m(m\pm1)}$

$$[j_1(j_1+1) + j_2(j_2+1) - j(j+1) + 2i(m-i)]c_i + \alpha^+(j_1, i-1)\alpha^-(j_2, m-i+1)c_{i-1} + \alpha^-(j_1, i+1)\alpha^+(j_2, m-i-1)c_{i+1} = 0$$

$$\left.\begin{aligned} j_1(j_1+1) &= \tfrac{1}{2}\left(\tfrac{1}{2}+1\right) = \tfrac{3}{4} \\ j_2(j_2+1) &= \tfrac{1}{2}\left(\tfrac{1}{2}+1\right) = \tfrac{3}{4} \end{aligned}\right\} \tfrac{3}{4} + \tfrac{3}{4} = \tfrac{3}{2}$$

pro $i = -\frac{1}{2}$:
$$\left[\frac{3}{2} - j(j+1) + 2\cdot\left(-\frac{1}{2}\right)\cdot\left(0-\left(-\frac{1}{2}\right)\right)\right]c_{-\frac{1}{2}} + \overbrace{\alpha^+\left(\tfrac{1}{2}, -\tfrac{3}{2}\right)\alpha^-\left(\tfrac{1}{2}, 0+\tfrac{3}{2}\right)}^{=0} c_{-\frac{3}{2}} + \underbrace{\alpha^-\left(\tfrac{1}{2}, \tfrac{1}{2}\right)\alpha^+\left(\tfrac{1}{2}, -\tfrac{1}{2}\right)}_{=1} c_{\frac{1}{2}} = 0$$

$$[1 - j(j+1)]c_{-\frac{1}{2}} + c_{\frac{1}{2}} = 0$$

pro $i = +\frac{1}{2}$:
$$\left[\frac{3}{2} - j(j+1) + 2\cdot\frac{1}{2}\left(0-\frac{1}{2}\right)\right]c_{\frac{1}{2}} + \overbrace{\alpha^+\left(\tfrac{1}{2}, -\tfrac{1}{2}\right)\alpha^-\left(\tfrac{1}{2}, -\tfrac{1}{2}\right)}^{=1} c_{-\frac{1}{2}} + \overbrace{\alpha^-\left(\tfrac{1}{2}, \tfrac{3}{2}\right)\alpha^+\left(\tfrac{1}{2}, -\tfrac{3}{2}\right)}^{0} c_{\frac{3}{2}} = 0$$

$$[1 - j(j+1)]\, c_{\frac{1}{2}} + c_{-\frac{1}{2}} = 0$$

+ normalizace $\quad |c_{\frac{1}{2}}|^2 + |c_{-\frac{1}{2}}|^2 = 1$

6. vyřešíme soustavu rovnic pro jednotlivá $j = 0, 1$:

- $j=0$: $\left.\begin{aligned}[1-0\cdot(0+1)]c_{-\frac{1}{2}} + c_{\frac{1}{2}} = 0\\ [1-0\cdot(0+1)]c_{\frac{1}{2}} + c_{-\frac{1}{2}} = 0\end{aligned}\right\} c_{\frac{1}{2}} = -c_{-\frac{1}{2}}$

  + norm. $|c_{\frac{1}{2}}|^2 + |c_{-\frac{1}{2}}|^2 = 1$

  $\Rightarrow \quad c_{\frac{1}{2}} = -c_{-\frac{1}{2}} = \frac{1}{\sqrt{2}}$

- $j=1$: $\left.\begin{aligned}[1-1\cdot(1+1)]c_{-\frac{1}{2}} + c_{\frac{1}{2}} = 0\\ [1-1\cdot(1+1)]c_{\frac{1}{2}} + c_{-\frac{1}{2}} = 0\end{aligned}\right\} c_{\frac{1}{2}} = c_{-\frac{1}{2}}$

  + normalizace

  $\Rightarrow \quad c_{\frac{1}{2}} = c_{-\frac{1}{2}} = \frac{1}{\sqrt{2}}$

7. napíšeme lineární kombinace

$j=1$: ~~$|1,0\rangle = \frac{1}{\sqrt{2}}(|+\rangle|+\rangle = 1$~~ ~~$|1,0\rangle = c_{\frac{1}{2}}|\frac{1}{2},\frac{1}{2}\rangle + c_{-\frac{1}{2}}|\frac{1}{2},-\frac{1}{2}\rangle$~~

~~$|1,0\rangle = \frac{1}{\sqrt{2}}(c_+$~~ ~~$|1,0\rangle = \frac{1}{\sqrt{2}}(|\frac{1}{2},\frac{1}{2}\rangle + |\frac{1}{2},-\frac{1}{2}\rangle) = \frac{1}{\sqrt{2}}(|+$~~

$|1,0\rangle = \frac{1}{\sqrt{2}}\left(|+\rangle|-\rangle + |-\rangle|+\rangle\right)$

$j=0$: $|0,0\rangle = \frac{1}{\sqrt{2}}\left(|+\rangle|-\rangle - |-\rangle|+\rangle\right)$

(V)

$S_x = \frac{1}{2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$ (viz dřívější cvičení)

$\rightarrow$ vl. čísla $\lambda_+ = \frac{1}{2} \quad |+\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$

$\lambda_- = -\frac{1}{2} \quad |-\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}$

$\Rightarrow |s_x = +\frac{1}{2}\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\end{pmatrix} + \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\end{pmatrix} = \underline{\underline{\frac{1}{\sqrt{2}}(|\uparrow\rangle + |\downarrow\rangle)}}$

(VI)

$|s_z = \frac{1}{2}, s_x = \frac{1}{2}\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\uparrow\rangle + |\uparrow\downarrow\rangle\right) = \frac{1}{\sqrt{2}}\left(|\uparrow\rangle|\uparrow\rangle + |\uparrow\rangle|\downarrow\rangle\right)$

$|0\,0\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\right)$

$\underline{\underline{\langle 0\,0|s_z = \frac{1}{2}, s_x = \frac{1}{2}\rangle}} = \frac{1}{\sqrt{2}}\left(\langle\uparrow\downarrow| - \langle\downarrow\uparrow|\right)\frac{1}{\sqrt{2}}\left(|\uparrow\uparrow\rangle + |\uparrow\downarrow\rangle\right) =$

$= \frac{1}{2}\Big[\underbrace{\langle\uparrow|\uparrow\rangle_1\langle\downarrow|\uparrow\rangle_2}_{0} - \underbrace{\langle\downarrow|\uparrow\rangle_1\langle\uparrow|\uparrow\rangle_2}_{0} + \underbrace{\langle\uparrow|\uparrow\rangle_1}_{1}\underbrace{\langle\downarrow|\downarrow\rangle_2}_{1} - \underbrace{\langle\downarrow|\uparrow\rangle_1}_{0}\underbrace{\langle\uparrow|\downarrow\rangle_2}_{0}\Big] = \underline{\underline{\frac{1}{2}}}$

$\Rightarrow \underline{\underline{|\langle 00|s_z = \frac{1}{2}, s_x = \frac{1}{2}\rangle|^2 = \frac{1}{4}}}$

VII $|l,m_l\rangle$ pro $l=1$: $|1,+1\rangle, |1,0\rangle, |1,-1\rangle \Rightarrow$ 3 stavy
$|s,m_s\rangle$ pro $s=\frac{1}{2}$: $|\frac{1}{2},+\frac{1}{2}\rangle, |\frac{1}{2},-\frac{1}{2}\rangle \Rightarrow$ 2 stavy
$\Big\}$ $3\cdot 2 = \underline{6}$ možností celkem

VIII
$$L^2|l,m_l\rangle = l(l+1)|l,m_l\rangle$$
$$L_z|l,m_l\rangle = m_l|l,m_l\rangle$$
$$S^2|s,m_s\rangle = s(s+1)|s,m_s\rangle$$
$$S_z|s,m_s\rangle = m_s|s,m_s\rangle$$
$$J^2|j,m_j\rangle = j(j+1)|j,m_j\rangle$$
$$J_z|j,m_j\rangle = m_j|j,m_j\rangle$$

l, s, j = velikost momentu hybnosti

m_l, m_s, m_j = projekce momentu hybnosti

IX $J_z|j,m_j\rangle = (L_z+S_z)|l,m_l\rangle|s,m_s\rangle = L_z|l,m_l\rangle|s,m_s\rangle + |l,m_l\rangle S_z|s,m_s\rangle = (m_l+m_s)|l,m_l\rangle|s,m_s\rangle$
$= m_j|j,m_j\rangle$

resp. stejně pro lib. lineární kombinaci stavů $\{|l,m_l\rangle|s,m_s\rangle\}$

$\Rightarrow$ všechny stavy jsou vl. stavy J_z

$\rightarrow$ hledáme LK takové, aby byly vl. stavy J^2

$J^2|j,m_j\rangle$: $\quad J^2 = L^2+S^2+(L_+S_-+L_-S_+)+2L_zS_z$

$$J^2|1,+1\rangle|\tfrac12,+\tfrac12\rangle = 2|1,+1\rangle|\tfrac12,+\tfrac12\rangle + \tfrac34|1,+1\rangle|\tfrac12,+\tfrac12\rangle + 0 + 0 + 2\cdot1\cdot\tfrac12|1,+1\rangle|\tfrac12,+\tfrac12\rangle =$$
$$= \tfrac{15}{4}|1,+1\rangle|\tfrac12,+\tfrac12\rangle = \tfrac32(\tfrac32+1)|1,+1\rangle|\tfrac12,+\tfrac12\rangle \Rightarrow j=\tfrac32$$
$$J^2|1,-1\rangle|\tfrac12,-\tfrac12\rangle = \dots = \tfrac32(\tfrac32+1)|1,-1\rangle|\tfrac12,-\tfrac12\rangle \Rightarrow j=\tfrac32$$
$$J^2|1,+1\rangle|\tfrac12,-\tfrac12\rangle = \tfrac{11}{4}|1,+1\rangle|\tfrac12,-\tfrac12\rangle + 0 + \sqrt2|1,0\rangle|\tfrac12,+\tfrac12\rangle + 2\cdot1\cdot(-\tfrac12)|1,+1\rangle|\tfrac12,-\tfrac12\rangle =$$
$$= \tfrac74|1,+1\rangle|\tfrac12,-\tfrac12\rangle + \sqrt2|1,0\rangle|\tfrac12,+\tfrac12\rangle$$
$$J^2|1,-1\rangle|\tfrac12,+\tfrac12\rangle = \tfrac74|1,-1\rangle|\tfrac12,+\tfrac12\rangle + \sqrt2|1,0\rangle|\tfrac12,-\tfrac12\rangle$$
$$J^2|1,0\rangle|\tfrac12,+\tfrac12\rangle = \tfrac{11}{4}|1,0\rangle|\tfrac12,+\tfrac12\rangle + \sqrt2|1,1\rangle|\tfrac12,-\tfrac12\rangle$$
$$J^2|1,0\rangle|\tfrac12,-\tfrac12\rangle = \tfrac{11}{4}|1,0\rangle|\tfrac12,-\tfrac12\rangle + \sqrt2|1,-1\rangle|\tfrac12,+\tfrac12\rangle$$

$\Rightarrow$ "správné" LK: $\quad |j,m_j\rangle$

$$|\tfrac32,+\tfrac32\rangle = |1,+1\rangle|\tfrac12,+\tfrac12\rangle$$
$$|\tfrac32,+\tfrac12\rangle = \sqrt{\tfrac23}|1,0\rangle|\tfrac12,+\tfrac12\rangle + \sqrt{\tfrac13}|1,1\rangle|\tfrac12,-\tfrac12\rangle \qquad |\tfrac12,+\tfrac12\rangle = \sqrt{\tfrac13}|1,0\rangle|\tfrac12,+\tfrac12\rangle - \sqrt{\tfrac23}|1,1\rangle|\tfrac12,-\tfrac12\rangle$$
$$|\tfrac32,-\tfrac12\rangle = \sqrt{\tfrac23}|1,0\rangle|\tfrac12,-\tfrac12\rangle + \sqrt{\tfrac13}|1,-1\rangle|\tfrac12,+\tfrac12\rangle \qquad |\tfrac12,-\tfrac12\rangle = \sqrt{\tfrac13}|1,0\rangle|\tfrac12,-\tfrac12\rangle - \sqrt{\tfrac23}|1,-1\rangle|\tfrac12,+\tfrac12\rangle$$
$$|\tfrac32,-\tfrac32\rangle = |1,-1\rangle|\tfrac12,-\tfrac12\rangle$$

X viz cvičení

XI viz cvičení