ÚDKM cv#7 DÚ

(1) $\hat{a}^+ = \frac{1}{\sqrt{2}}(\hat{x} - i\hat{p})$

$\hat{a} = \frac{1}{\sqrt{2}}(\hat{x} + i\hat{p})$

• základní stav:

$a|0\rangle = 0$

$\frac{1}{\sqrt{2}}(\hat{x} + i\hat{p})|0\rangle = 0$

$\downarrow$ v souř. repre

$\frac{1}{\sqrt{2}}\left(x + i\left(-i\frac{d}{dx}\right)\right)\psi_0(x) = 0$

$\left(x + \frac{d}{dx}\right)\psi_0(x) = 0$

$\frac{d\psi_0(x)}{dx} = -x\psi_0(x)$

$\int \frac{1}{\psi_0} d\psi_0 = -\int x\,dx$

$\ln \psi_0 = -\frac{1}{2}x^2 + C$

$\psi_0 = N e^{-\frac{1}{2}x^2}$

z normalizace:

$\langle\psi_0|\psi_0\rangle = \int_{-\infty}^{+\infty} \psi_0^*(x)\psi_0(x)\,dx \overset{!}{=} 1$

$|N|^2 \int_{-\infty}^{+\infty} e^{-x^2} dx \overset{!}{=} 1$

$|N|^2 \sqrt{\pi} = 1$

$|N| = \frac{1}{\sqrt[4]{\pi}}$

$$\Rightarrow \boxed{\text{zákl. stav} \quad |0\rangle = \frac{1}{\sqrt[4]{\pi}} e^{-\frac{1}{2}x^2}}$$

- první excitovaný stav

$$a^+|0\rangle = |1\rangle \qquad \left(a^+|n\rangle = \sqrt{n+1}\,|n+1\rangle \right)$$

$$\Rightarrow |1\rangle = \frac{1}{\sqrt{2}}(\hat{x} - i\hat{p})|0\rangle =$$

$$\Rightarrow \psi_1(x) = \frac{1}{\sqrt{2}}\left(x - i\left(-i\frac{d}{dx}\right)\right)\psi_0(x) =$$

$$= \frac{1}{\sqrt{2}}\left(x - \frac{d}{dx}\right)\frac{1}{\sqrt[4]{\pi}} e^{-\frac{1}{2}x^2} =$$

$$= \frac{1}{\sqrt{2}}\frac{1}{\sqrt[4]{\pi}}\left(x e^{-\frac{1}{2}x^2} - e^{-\frac{1}{2}x^2}\cdot(-x)\right) =$$

$$= \sqrt[4]{\frac{4}{\pi}}\, x e^{-\frac{1}{2}x^2}$$

$$\boxed{\psi_1(x) = \sqrt[4]{\frac{4}{\pi}}\, x e^{-\frac{1}{2}x^2}}$$

- druhý excitovaný stav

$$a^+|1\rangle = \sqrt{2}\,|2\rangle$$

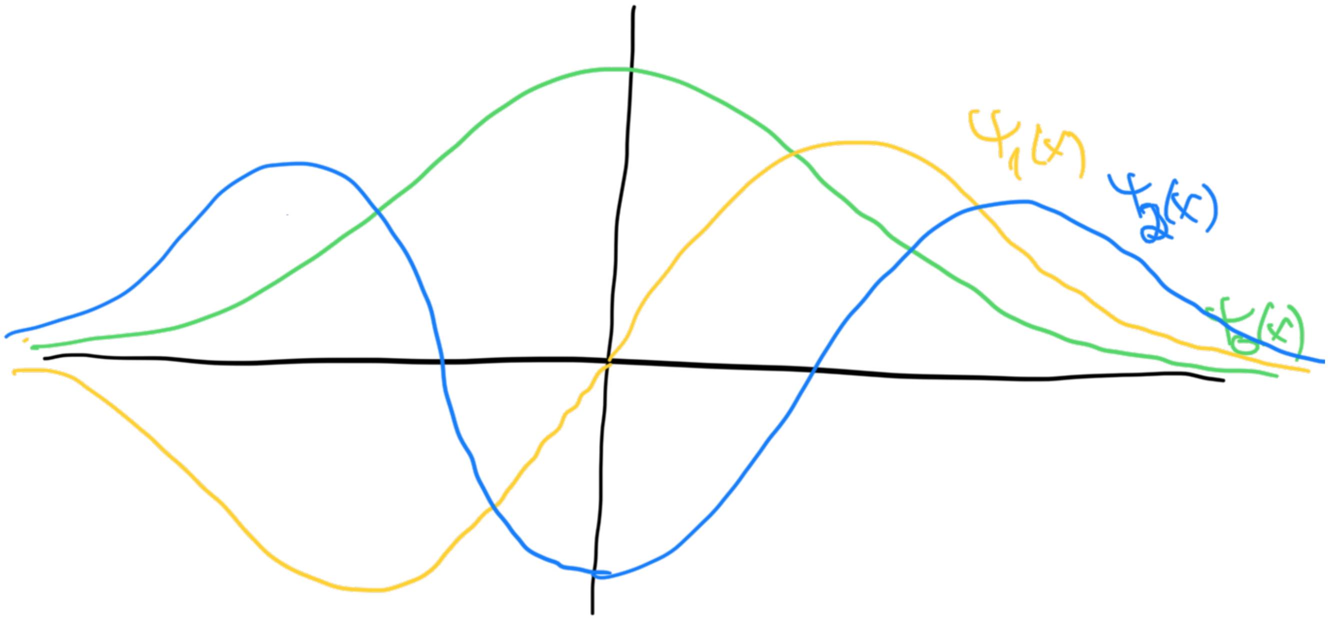
$$\Rightarrow |2\rangle = \frac{1}{2}(\hat{x} - i\hat{p})|1\rangle$$

$$\Rightarrow \psi_2(x) = \frac{1}{2}\left(x - i\left(-i\frac{d}{dx}\right)\right)\sqrt[4]{\frac{4}{\pi}}\, x e^{-\frac{1}{2}x^2} =$$

$$= \frac{1}{\sqrt[4]{4\pi}}\left(x^2 e^{-\frac{1}{2}x^2} - e^{-\frac{1}{2}x^2} - x e^{-\frac{1}{2}x^2}(-x)\right) =$$

$$= \frac{1}{\sqrt[4]{4\pi}}(2x^2 - 1)e^{-\frac{1}{2}x^2}$$

$$\boxed{\psi_2(x) = \sqrt[4]{4\pi}\,(2x^2-1)\,e^{-2x^2}}$$



② $\hat{J}_0 = \frac{1}{2}(a_2^\dagger a_2 - a_1^\dagger a_1)$

$\hat{J}_+ = a_2^\dagger a_1$

$\hat{J}_- = a_1^\dagger a_2$

$[a_i, a_j^\dagger] = \delta_{ij} \qquad [a_i, a_j] = 0 \qquad [a_i^\dagger, a_j^\dagger] = 0$

$[AB, C] = A[B,C] + [A,C]B$

$[A, BC] = B[A,C] + [A,B]C$

$\underline{\underline{[J_0, J_+]}} = \left[\frac{1}{2}(a_2^\dagger a_2 - a_1^\dagger a_1), a_2^\dagger a_1\right] =$

$= \frac{1}{2}\left([a_2^\dagger \underbrace{a_2, a_2^\dagger}\, a_1] - [a_1^\dagger a_1, a_2^\dagger a_1]\right) =$

$= \frac{1}{2}\left(\ldots + [a_2, a_2^\dagger] a_1 - [a_1^\dagger, a_1] a_1 a_2^\dagger \right) =$

$$= \frac{1}{2}\Big(a_2^{+}\underbrace{[a_2, a_2^{+}]}_{=1}a_1 \quad [illegible] \underbrace{\qquad}_{=-1} \quad [illegible]\Big)$$

$$= \frac{1}{2}\left(a_2^{+}a_1 + a_1 a_2^{+}\right) =$$

$$= a_2^{+}a_1 =$$

$$= \underline{\underline{J_+}}$$

$$[J_0, J_-] = \left[\frac{1}{2}(a_2^{+}a_2 - a_1^{+}a_1), a_1^{+}a_2\right] =$$

$$= \frac{1}{2}\left([a_2^{+}a_2, a_1^{+}a_2] - [a_1^{+}a_1, a_1^{+}a_2]\right) =$$

$$= \frac{1}{2}\Big(a_2 a_1^{+}\underbrace{[a_2^{+}, a_2]}_{=-1} - a_1^{+}a_2\underbrace{[a_1, a_1^{+}]}_{=1}\Big) =$$

$$= \frac{1}{2}\left(-a_2 a_1^{+} - a_1^{+}a_2\right) =$$

$$= -a_1^{+}a_2 =$$

$$= \underline{\underline{-J_-}}$$

$$\underline{\underline{[J_+, J_-]}} = [a_2^{+}a_1, a_1^{+}a_2] =$$

$$= a_2^{+}\underbrace{[a_1, a_1^{+}]}_{=1}a_2 + a_1^{+}\underbrace{[a_2^{+}, a_2]}_{=-1}a_1 =$$

$$= a_2^{+}a_2 - a_1^{+}a_1 =$$

$$= \underline{\underline{2J_0}}$$