DVA OSCILÁTORY STEJNÉ FREKVENCE ②

$$\hat{H} = \hat{H}_1 + \hat{H}_2 = \frac{1}{2}(\hat{P}_1^2 + \hat{Q}_1^2) + \frac{1}{2}(\hat{P}_2^2 + \hat{Q}_2^2) \qquad [\hat{P}_i, \hat{Q}_j] = i\delta_{ij}$$

(IV) $\hat{N}_1 = a_1^+ a_1$, $\hat{N}_2 = a_2^+ a_2$ → operátory počtu částic
(počtu excitací)

$$\underline{\underline{N|n\rangle}} = a^+ a|n\rangle = a^+ \sqrt{n}\,|n-1\rangle = \sqrt{n}\,a^+|n-1\rangle = \sqrt{n}\cdot\sqrt{n}\,|n\rangle = \underline{\underline{n|n\rangle}}$$

$$a^+|n\rangle = \sqrt{n+1}\,|n+1\rangle$$
$$a|n\rangle = \sqrt{n}\,|n-1\rangle$$

(V)
$$\underline{\hat{H}} = \frac{1}{2}(P_1^2 + Q_1^2) + \frac{1}{2}(P_2^2 + Q_2^2)$$
$$= \hbar\omega(a_1^+ a_1 + \tfrac{1}{2}) + \hbar\omega(a_2^+ a_2 + \tfrac{1}{2})$$
$$= \underline{\underline{\hbar\omega(\hat{N}_1 + \hat{N}_2 + 1)}}$$

$$\underline{\underline{E}} = \langle n|\hat{H}|n\rangle = \langle n|\hbar\omega(\hat{N}_1 + \hat{N}_2 + 1)|n\rangle = \underline{\underline{\hbar\omega(n_1 + n_2 + 1)}}$$

hladiny degenerovaní
~~$(n_1, n_2$ ...~~
$n = n_1 + n_2$ → důležitý součet

(VI)
$$\hat{J}_0 = \frac{1}{2}(a_2^+ a_2 - a_1^+ a_1)$$
$$\hat{J}_+ = \hat{a}_2^+ \hat{a}_1$$
$$\hat{J}_- = \hat{a}_1^+ a_2$$

$$\underline{\underline{[\hat{J}_0, \hat{H}]}} = \left[\frac{1}{2}(a_2^+ a_2 - a_1^+ a_1), \hbar\omega(a_1^+ a_1 + a_2^+ a_2 + 1)\right] = \qquad [a_i^+, a_j^+] = \delta_{ij}$$

$$= \frac{\hbar\omega}{2}[\underbrace{a_2^+ a_2}_{N_2}, \underbrace{a_2^+ a_2}_{N_2}] - \frac{\hbar\omega}{2}[\underbrace{a_1^+ a_1}_{N_1}, \underbrace{a_1^+ a_1}_{N_1}] =$$

← ostatní členy spolu zjevně komutují
$[\hat{N}_i, \hat{N}_j] = 1 - \delta_{ij}$

$$\underline{\underline{= 0}}$$

pozor na pořadí! A, B, C

$$[AB, C] = A[B, C] + [A, C]B$$

$$\underline{\underline{[\hat{J}_+, \hat{H}]}} = [a_2^+ a_1, \hbar\omega(a_1^+ a_1 + a_2^+ a_2 + 1)] =$$
$$= \hbar\omega[a_2^+ a_1, a_1^+ a_1] + \hbar\omega[a_2^+ a_1, a_2^+ a_2] =$$
$$= \hbar\omega\, a_2^+ \underbrace{[a_1, a_1^+ a_1]}_{} + \hbar\omega \underbrace{[a_2^+, a_1^+ a_1]}_{=0} a_1 + \hbar\omega\, a_2^+ \underbrace{[a_1, a_2^+ a_2]}_{=0} + \hbar\omega \underbrace{[a_2^+, a_2^+ a_2]}_{} a_1$$

$$a_1^+ \underbrace{[a_1, a_1]}_{=0} + \underbrace{[a_1, a_1^+]}_{=1} a_1 \qquad = a_2^+ \underbrace{[a_2^+, a_2]}_{-1} + \underbrace{[a_2^+, a_2^+]}_{=0} a_2$$

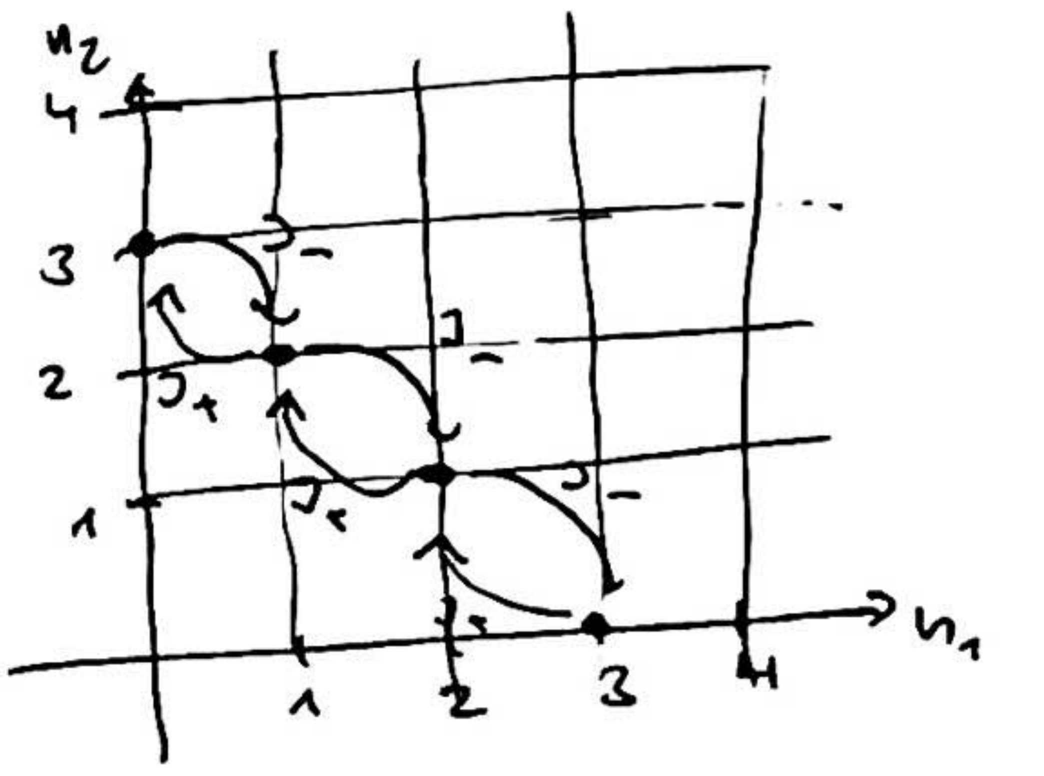
$$= \hbar\omega\, a_2^+ a_1 + \hbar\omega(-a_2^+) a_1 = \underline{\underline{0}}$$

a podobně $\underline{\underline{[J_-, H] = 0}}$

(VII) zcela analogicky

$$[\hat{J}_0, \hat{J}_+] = \hat{J}_+$$
$$[\hat{J}_0, \hat{J}_-] = -\hat{J}_-$$
$$\underline{\underline{[\hat{J}_+, \hat{J}_-] = 2\hat{J}_0}}$$

VIII



$H|n_1 n_2\rangle = (n_1+n_2+1)|n_1 n_2\rangle$

$J_0|n_1 n_2\rangle = \left[\frac{1}{2}(a_2^+a_2 - a_1^+a_1)\right]|n_1 n_2\rangle = \frac{1}{2}(n_2-n_1)|n_1 n_2\rangle$

$J_+|n_1 n_2\rangle = a_2^+a_1|n_1 n_2\rangle = \sqrt{n_1}\sqrt{n_2+1}\,|n_1-1, n_2+1\rangle$

$J_-|n_1 n_2\rangle = a_1^+a_2|n_1 n_2\rangle = \sqrt{n_1+1}\sqrt{n_2}\,|n_1+1, n_2-1\rangle$

IX

$n_1 n_2 \to j, m$ (jiné značení stavu)

$J^2|j,m\rangle = j(j+1)|j,m\rangle$

$J_0|j,m\rangle = m|j,m\rangle$

A_3B_3

$A\cdot B = \frac{1}{2}(A_+B_- + A_-B_+) + A_0B_0$

$J^2|n_1 n_2\rangle = \left[\frac{1}{2}(J_+J_- + J_-J_+) + J_0^2\right]|n_1 n_2\rangle =$

$= \left\{\frac{1}{2}\left[(n_1+1)n_2 + (n_2+1)n_1\right] + \frac{1}{4}(n_2-n_1)^2\right\}|n_1 n_2\rangle =$

$= \dots = \left(\frac{n_1+n_2}{2}\right)\left(\frac{n_1+n_2}{2}+1\right)|n_1 n_2\rangle$

$J_0|n_1 n_2\rangle = \frac{1}{2}(n_2-n_1)|n_1 n_2\rangle$

$\Rightarrow$ identifikujeme $m = \frac{1}{2}(n_2-n_1)$

$\underline{\underline{j = \frac{1}{2}(n_1+n_2)}}$

$\Rightarrow \underline{\underline{E}} = n_1+n_2+1 = \underline{\underline{2j+1}}$

LHO: PST VÝSKYTU ČÁSTICE V „ZAKÁZANÉ" OBLASTI

LHO: $\hat{H} = \hat{T} + \hat{V} = \frac{\hat{p}^2}{2m} + \frac{1}{2} m\omega^2 \hat{x}^2$

(přeškálované $\hat{H} = \frac{\hat{p}^2}{2} + \frac{\hat{x}^2}{2}$) → $\psi_0(x) = \sqrt[4]{\frac{1}{\pi}}\, e^{-\frac{x^2}{2}}$, $E_0 = \frac{1}{2}$

→ fce zákl. stavu: $\psi_0(x) = \sqrt[4]{\frac{m\omega}{\pi\hbar}}\, e^{-\frac{x^2}{2}\cdot\frac{m\omega}{\hbar}}$ → $\psi_0(x) = \left(\frac{1}{\pi x_s^2}\right)^{\frac{1}{4}} e^{-\frac{x^2}{2x_s^2}}$

$x_s = \sqrt{\frac{\hbar}{m\omega}}$ = typická škála

→ energie zákl. stavu: $E_0 = \frac{1}{2}\hbar\omega$

(A) KLASICKY:

rovnovážná poloha $x_0 = 0$

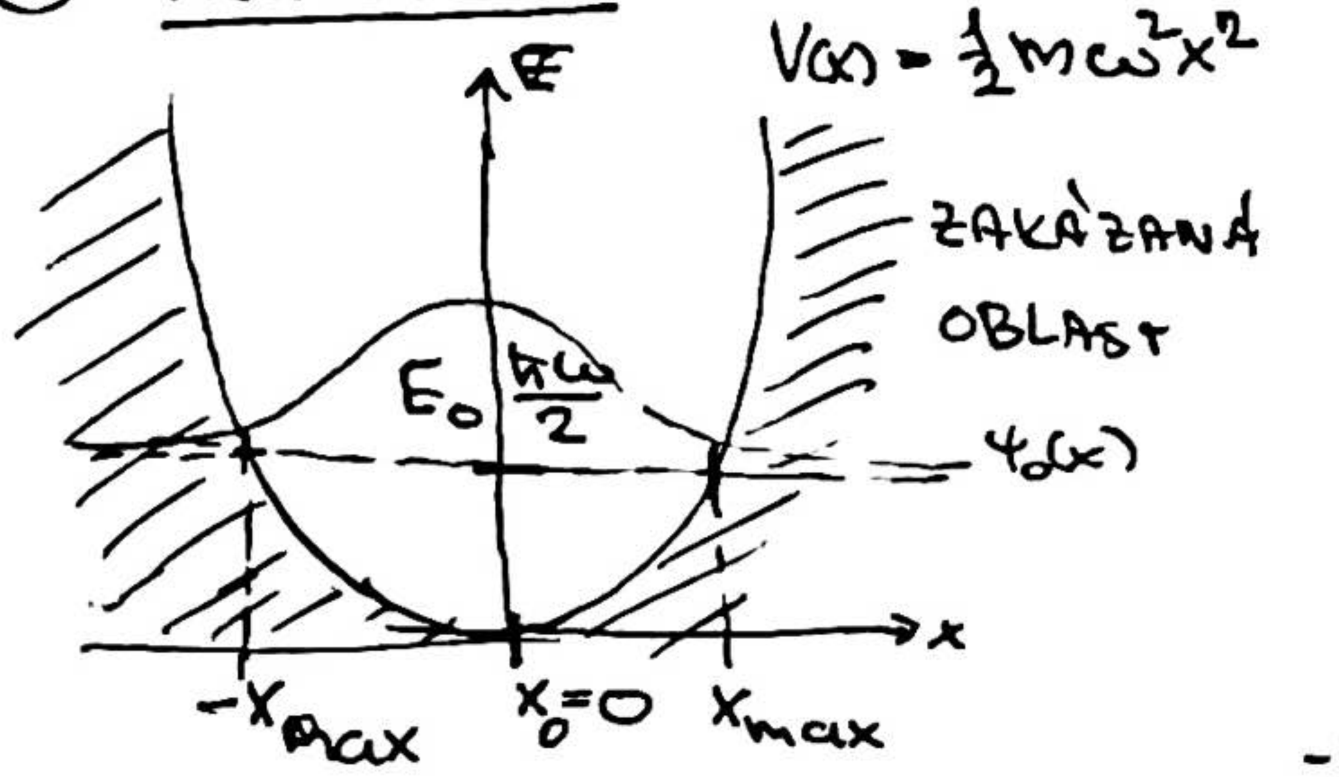
a částice se může vychýlit do polohy x_{max} takové, že $V(x_{max})$ = tot. energie

$$\frac{1}{2} m\omega^2 x_{max}^2 = \frac{1}{2}\hbar\omega \qquad\qquad \frac{1}{2}x_{max}^2 = \frac{1}{2}$$

$$x_{max} = \pm 1$$

$$\Rightarrow x_{max} = \pm\sqrt{\frac{\hbar}{m\omega}}$$

(B) KVANTOVĚ:



pst. výskytu za x_{max}: $P = \int_{-\infty}^{-x_{max}} |\psi_0(x)|^2 dx + \int_{x_{max}}^{+\infty} |\psi_0(x)|^2 dx = 2\int_{x_{max}}^{+\infty} |\psi_0(x)|^2 dx$

$$= 2\int_{x_{max}}^{+\infty} \frac{m\omega}{\hbar\sqrt{\pi}} e^{-x^2\frac{m\omega}{\hbar}} dx = \quad \text{subst. } x\sqrt{\frac{m\omega}{\hbar}} = t \quad dx = \sqrt{\frac{\hbar}{m\omega}}\,dt \quad x_{max} \Rightarrow 1$$

$$= \frac{2}{\sqrt{\pi}} \int_{1}^{+\infty} e^{-t^2} dt = (*)$$

$$= \frac{2}{\sqrt{\pi}} \cdot 0{,}16\,\frac{\sqrt{\pi}}{2} \doteq \underline{\underline{0{,}16}} \quad \text{tj. } \underline{\underline{16\,\%}}$$

(*) buď software, nebo:

$$\int_0^{\infty} e^{-t^2} dt = \frac{1}{2}\underbrace{\int_{-\infty}^{\infty} e^{-t^2} dt}_{=\sqrt{\pi}} = \frac{1}{2}\sqrt{\pi}$$

$$\int_1^{\infty} e^{-t^2} dt = \underbrace{\int_0^{\infty} e^{-t^2} dt}_{\frac{\sqrt{\pi}}{2}} - \underbrace{\int_0^{1} e^{-t^2} dt}_{0{,}84\cdot\frac{\sqrt{\pi}}{2}} = \frac{1}{2}\sqrt{\pi} - 0{,}84\,\frac{\sqrt{\pi}}{2} = 0{,}16\,\frac{\sqrt{\pi}}{2}$$

numericky nebo software